DES 2023 Final Exam Solutions

(1)(a) $x^3y'' - 4xy' + 4y = 0$. $y_1 = x$ is a solution

Now $y'' - \frac{4}{x^2}y' + \frac{4}{x^3}y = 0$

$p(x) = -\frac{4}{x^2}$. $\int p(x)dx = \frac{4}{x}$.

$\therefore y_2 = y_1 \int \frac{e^{-\frac{4}{x}}}{x^2} dx$. Then put $u = \frac{1}{x}$.

$du = \frac{-1}{x^2} dx$.

$$\int \frac{e^{-4/x}}{x^2} dx = -\int e^{-4u} du = \frac{1}{4} e^{-4u} = \frac{1}{4} e^{-\frac{4}{x}},$$

So $y_2 = \frac{x}{4} e^{-\frac{4}{x}}$.

or we can take $y_2 = x e^{-\frac{4}{x}}$.

(b) $y'' - 4y' + 4y = e^x$. First $y'' - 4y' + 4y = 0$ has auxilliary equation

$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$

So solutions are $y_1 = e^{2x}$, $y_2 = xe^{2x}$.

$y_h = c_1 e^{2x} + c_2 x e^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}.$$

$R(x) = e^x$. A particular solution is

$y_p = uy_1 + vy_2$

$$u = -\int \frac{y_2 R}{W} = -\int \frac{e^x (xe^{2x})}{e^{4x}} dx$$

$$= -\int xe^{-x} dx = +xe^{-x} - \int e^{-x} dx$$

$$= (x+1)e^{-x}.$$

$$v = \int \frac{y_1 R}{W} = \int \frac{e^{2x} e^x}{e^{4x}} dx = \int e^{-x} dx = -e^{-x}.$$

$\therefore y_p = e^{2x}((x+1)e^{-x}) - (xe^{2x})e^{-x} = e^x$

$y = e^x + y_h$.

Q2. $y'' + 4xy' + y = 0 \qquad y = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 + a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty}(4n+1)a_n x^n$$

$$= 2a_2 + a_0 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}(4n+1)a_n x^n$$

$$= 2a_2 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (4n+1)a_n\right]x^n = 0$$

$$a_2 = -\tfrac{1}{2}a_0$$

and $a_{n+2} = \dfrac{-(4n+1)a_n}{(n+2)(n+1)}, \quad n \geq 1$

Even terms

Let gen den be $an+b$

$n=2. \quad a_4 = \dfrac{-9}{4.3}a_2 = \dfrac{9}{2(3\times4)}a_0$

$4a+b = 25$

$n=4 \quad a_6 = \dfrac{-17}{6.5}a_4 = \dfrac{(-1)^3\, 9.17}{2.(3\times4\times5\times6)}a_0$

$3a+b = 17$

$a = 8$

$b = -7$

$n=6 \quad a_8 = \dfrac{-25a_6}{8\times7} = \dfrac{(-1)^4(1\times9\times17\times25)}{8!}a_0$

$$a_{2n} = \frac{(-1)^n(1\times9\times\ldots\times(8n-7))}{(2n)!}a_0, \quad n \geq 1$$

$n=1 \quad a_3 = -\dfrac{5a_1}{3\times2}$

$n=3 \quad a_5 = \dfrac{-13a_3}{5\times4} = \dfrac{(-1)^2(5\times13)a_1}{1\times2\times3\times4\times5}$

$3a+b = 21$

$2a+b = 13$

$n=5 \quad a_7 = \dfrac{(-1)^3\,21a_5}{7\times6} = \dfrac{(-1)^3(5\times13\times21)a_1}{7!}$

$a = 8$

$b = -3$

$$a_{2n+1} = \frac{(-1)^n(5\times13\times\ldots\times(8n-3))a_1}{(2n+1)!}$$

$$y = a_0\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n(1\times9\times\ldots\times(8n-7))x^{2n}}{(2n)!}\right) + a_1\left(x + \sum_{n=1}^{\infty}(-1)^n\frac{(5\times13\times\ldots\times(8n-3))x^{2n+1}}{(2n+1)!}\right)$$

23(a) $x^2y'' + 3xy' + \left(x + \frac{8}{9}\right)y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + 3\sum_{n=0}^{\infty}(n+s)a_n x^{n+s}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+s+1} + \frac{8}{9}\sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\doteq \left(s(s-1)+3s+\frac{8}{9}\right)a_0 x^s + \sum_{n=1}^{\infty}\left\{(n+s)(n+s-1)+3(n+s)+\frac{8}{9}\right\}a_0 x^{s+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+s+1}$$

$$= \left(s^2+2s+\frac{8}{9}\right)a_0 x^s + \sum_{n=1}^{\infty}\left(\left[(n+s)(n+s+2)+\frac{8}{9}\right]a_n + a_{n-1}\right)x^{n+s} = 0$$

$\therefore\ s^2+2s+8/9 = 0 \qquad \therefore\ s = -\frac{4}{3}, -\frac{2}{3}$

$$a_n = \frac{-a_{n-1}}{(n+s)(n+s+2)+8/9} \qquad n \geq 1$$

$$= \frac{-9a_{n-1}}{9(n+s)(n+s+2)+8}$$

$S_1 = -\frac{4}{3}$

$$a_n = \frac{-9a_{n-1}}{9(n-\frac{4}{3})(n-\frac{4}{3}+2)+8} = \frac{-3a_{n-1}}{n(3n-2)}$$

So $a_1 = \frac{-3a_0}{1(3-2)} = 3a_0$

$$a_2 = \frac{-3a_1}{2(6-4)} = \frac{(-1)^2 9a_0}{(1\times2)(1\times4)}$$

$$a_3 = \frac{-3a_2}{3\times7} = \frac{(-1)^3 3^3 a_0}{(1\times2\times3)(1\times4\times7)}$$

$$a_4 = \frac{(-1)^4 3^4 a_0}{4!(1\times4\times.\times10)}$$

$$a_n = \frac{(-3)^n a_0}{n!(1\times4\cdots\times(3n-2))} \qquad n \geq 1.$$

$s = \frac{-2}{3}$ $\quad a_n = \frac{-9a_{n-1}}{9(n-\frac{2}{3})(n-\frac{2}{3}+2)+8}$

$= \frac{-3a_{n-1}}{n(3n+2)}, \quad n \geq 1.$

$a_1 = \frac{-3a_1}{1 \times 5}$

$a_2 = \frac{(-1)^2 3^2 a_0}{(1\times2)(5\times8)}$

$a_3 = \frac{(-1)^3 3^3 a_0}{(1\times2\times3)(5\times8\times11)}$ etc

$a_n = \frac{(-1)^n 3^n a_0}{n!(5\times \ldots \times(3n+2))}$

$y = A\sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{n-2/3}}{n!(5\times\ldots(3n+2))}$ is solution corresponding to $s = -2/3$

$y = B\sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{n-4/3}}{n!(1\times4\ldots\times(3n-2))}$ is solution corresponding to $s = -4/3$.

Now $1-2s = 3$, $a^2r^2x^{2r} = x$

$s^2 - r^2\alpha^2 = 8/9$. $s=-1$, $r = 1/2$, $a = 2$

$\alpha = \pm\frac{2}{3}$

So $y = c_1 \frac{1}{x} J_{-\frac{2}{3}}(2\sqrt{x}) + c_2 \frac{1}{x} J_{\frac{2}{3}}(2\sqrt{x})$

is solution

Q4 (i) $y'' + 3y' + 2y = e^{-3x}$ Take LT

$$(s^2+3s+2)Y(s) - sy(0) - y'(0) + 3y(0) = \frac{1}{s+3}$$

Now $y(0) = y'(0) = 0$

$$\therefore \quad Y(s) = \frac{1}{(s+3)(s^2+3s+2)}$$

$$= \frac{1}{(s+1)(s+2)(s+3)}$$

Now $$\frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{3+s}$$

$$\frac{1}{(s+2)(s+3)} = A + \frac{B(s+1)}{s+2} + \frac{C(s+1)}{s+3} \quad \text{put } s=-1$$

$A = \frac{1}{2}$.

$$\frac{1}{(s+1)(s+3)} = \frac{\frac{1}{2}(s+2)}{s+1} + B + \frac{C(s+2)}{s+3}. \quad s=-2$$

$B = -1$

$$\frac{1}{(s+1)(s+2)} = \frac{\frac{1}{2}(s+3)}{s+1} - \frac{\frac{1}{2}(s+3)}{s+1} + C, \quad s=-3$$

$C = \frac{1}{2}$.

$$\therefore \quad y = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{1}{s+1} - \frac{1}{1}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3}\right)$$

$$= \frac{1}{2}e^{-x} - e^{-2x} + \frac{1}{2}e^{-3x}.$$

(b) $\mathcal{L}(J_0(t)) = \frac{1}{\sqrt{s^2+1}}$, $\mathcal{L}(\sin t) = \frac{1}{s^2+1}$.

So $$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+1}}\frac{1}{s^2+1}\right) = \int_0^t \sin u \, J_0(t-u)\,du.$$

Q5(i) $b_n = 2\int_0^1 (1-x)\sin(n\pi x)\,dx$

$$= \left[-\frac{2\sin(n\pi x)}{\pi^2 n^2} + \frac{2x\cos(\pi n x)}{\pi n} - \frac{2\cos(\pi n x)}{\pi n}\right]_0^1$$

$$= \frac{2}{n\pi}.$$

(ii) $\frac{1}{4}u_t = u_{xx}$ $\quad 0 \le x \le 1, \quad t > 0$

$u(0,t) = u(1,t) = 0$

$u(x,0) = 1-x,$

Put $u(x,t) = X(x)T(t)$ $\quad X(0) = X(1) = 0.$

$$\frac{1}{4}\frac{T'}{T} = \frac{X''}{X} = \lambda, \quad \lambda \text{ a constant}$$

If $\lambda = k^2 > 0$ $\quad X'' = k^2 X$ $\quad \therefore X(x) = Ae^{kx} + Be^{-kx}$

$X(0) = X(1) = 0$

$A + B = 0 \Rightarrow A = B = 0$

$Ae^{k} + Be^{-k} = 0$

So $\lambda \neq k^2$

$\lambda = 0 \Rightarrow X = Ax + B$

$X(0) = B = 0. \quad X(1) = A = 0$

$\therefore \lambda \neq 0$

$\lambda = -k^2$ $\quad X(x) = A\cos(kx) + B\sin(kx) = 0$

$X(0) = A = 0$

$X(1) = B\sin(k) = 0$ $\quad \therefore k = n\pi$

$n = 1, 2, \ldots$

$\therefore \lambda = -n^2\pi^2$

$$\frac{1}{4}T' = -n^2\pi^2 T \Rightarrow T' = -4n^2\pi^2 T$$

$$T = Ce^{-4n^2\pi^2 t}$$

get solutions $u_n(x,t) = B_n e^{-4n^2\pi^2 t}\sin(n\pi x)$

Take a superposition to get

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-4\pi^2 n^2 t}\sin(n\pi x)$$

$$B_n = 2\int_0^1 u(x,0)\sin(n\pi x)\,dx = \frac{2}{n\pi}.$$

Thus $$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n^2\pi^2 t}\sin(n\pi x).$$