

DE Exam 2024 Solutions

①

1 (i) $x^3 y'' - 2xy' + 2y = 0$

or $y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = 0$

$y_1 = x$ is a solution.

$p(x) = -\frac{2}{x^2}$

$\int p(x) = -2 \int x^{-2} dx = \frac{2}{x}$

$e^{-\int p(x) dx} = e^{-\frac{2}{x}}$

$y_2 = y_1 \int \frac{e^{-\frac{2}{x}}}{x^2} dx$ put $-\frac{2}{x} = u$ (4)

$du = \frac{2}{x^2} dx \therefore \frac{dx}{x^2} = \frac{1}{2} du$

$y_2 = x \int \frac{1}{2} e^u du = \frac{1}{2} x e^u = \frac{1}{2} x e^{-\frac{2}{x}}$

(ii) $y'' - 4y' + 4y = e^{2x}$ $y_1 = e^{2x}$, $y_2 = x e^{2x}$

$R(x) = e^{2x}$

$W = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix}$ (6)

$= e^{4x} + 2x e^{4x} - 2x e^{4x}$
 $= e^{4x}$

$u' = \frac{-y_2 R}{W} = \frac{-x e^{2x} e^{2x}}{e^{4x}} = -x$ $u = -\frac{1}{2} x^2$

$v' = \frac{y_1 R}{W} = \frac{e^{2x} e^{2x}}{e^{4x}} = 1$ so $v = x$

$y_p = y_1 u + y_2 v = -\frac{1}{2} x e^{2x} + x e^{2x} = \frac{1}{2} x e^{2x}$

(2)

$$Q(2) \quad y'' - 2xy' - y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\text{So } \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} a_n x^n =$$

$$2a_2 - a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=1}^{\infty} a_n x^n$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} (2n+1)a_n x^n$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (2n+1)a_n] x^n = 0$$

$$\therefore a_2 = \frac{1}{2}a_0 \quad a_{n+2} = \frac{(2n+1)}{(n+2)(n+1)} a_n, \quad n \geq 1.$$

$$n=2, \quad a_4 = \frac{5}{4 \times 3} a_2 = \frac{5}{2 \times 3 \times 4} a_0$$

$$n=4 \quad a_6 = \frac{9 \times 5}{6 \times 5 \times 4 \times 3 \times 2} a_0$$

$$n=6 \quad a_8 = \frac{5 \times 9 \times 13}{8!} a_0$$

$$a_{2n} = \frac{5 \times \dots \times (4n-3)}{(2n)!} a_0$$

$$y_1 = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{5 \times \dots \times (4n-3)}{(2n)!} x^{2n} \right)$$

For numerator
last term is
 $a_n + b$

$$\begin{aligned} n=3 &\Rightarrow 3a+b=9 \\ n=4 &\Rightarrow 4a+b=13 \\ \text{So } a &= 4 \\ b &= -3 \end{aligned}$$

(3)

$$n=1, a_3 = \frac{3}{3 \times 2} a_1,$$

$$n=3, a_5 = \frac{7}{5 \times 4} a_3 = \frac{3 \times 7}{5!} a_1,$$

$$n=5, a_7 = \frac{3 \times 7 \times 11}{7!} a_1, \text{ etc}$$

$$a_{2n+1} = \frac{3 \times \dots \times (4n-1)}{(2n+1)!} a_1, \quad n \geq 1$$

$$y_2 = a_1 \left(x + \sum_{n=1}^{\infty} \frac{3 \times \dots \times (4n-1)}{(2n+1)!} x^{2n+1} \right)$$

$$\text{Q3} \quad xy'' + 2y' - y = 0 \quad y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + 2 \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= (s(s-1) + 2s) a_0 x^{s-1} + \sum_{n=1}^{\infty} [(n+s)(n+s-1) + 2(n+s)] a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= (s^2 + s) a_0 x^{s-1} + \sum_{n=1}^{\infty} \left([(n+s)(n+s-1) + 2(n+s)] a_n - a_{n-1} \right) x^{n+s-1}$$

$$= 0$$

$$s^2 + s = 0 \quad s = 0 \text{ or } -1$$

$$\text{If } s=0 \quad a_n = \frac{a_{n-1}}{n(n+1)}$$

$$a_n = \frac{a_0}{n!(n+1)!}$$

(4)

$$\text{So } y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!}$$

$$(ii) \quad x^2 y'' + 2xy' - xy = 0$$

$$1-2s=2, \quad s^2 - r^2 \alpha^2 = 0$$

$$s = -\frac{1}{2}, \quad a^2 r^2 x^{2r} = -x, \quad r = \frac{1}{2}$$

$$\frac{1}{4} - \frac{1}{4} \alpha^2 = 0 \quad \alpha = \pm 1$$

$$\text{Solution is } y = c_1 x^{-1/2} I_1\left(\frac{x^{1/2}}{2}\right) + c_2 x^{-1/2} K_1\left(\frac{x^{1/2}}{2}\right)$$

$$\text{Q4. (i) } y'' + 5y' + 6y = e^{-x}, \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}(y'' + 5y' + 6y) = \mathcal{L}(e^{-x}) \quad \mathcal{L}(y) = Y$$

$$(s^2 + 5s + 6)Y(s) = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

$$= \frac{1}{2} \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2(s+3)}$$

$$y(x) = \frac{1}{2} e^{-x} - e^{-2x} + \frac{1}{2} e^{-3x}$$

$$(ii) \quad F(s) = \frac{s}{(s^2+4)\sqrt{4+s^2}}$$

$$f(t) = \int_0^t \cos(2(t-u)) J_0(2u) du$$

(5)

$$Q5. (i) b_n = 2 \int_0^1 (1-x^2) \sin(n\pi x) dx$$

$$= \frac{4 + 2n^2\pi^2 - 4(-1)^n}{n^3\pi^3}$$

$$= \frac{4(1+(-1)^n)}{n^3\pi^3} + \frac{2}{n\pi}$$

$$(ii) \quad \frac{1}{2} u_t = u_{xx} \quad 0 < x \leq 1, \quad u(x,0) = f(x) \\ u = XT, \quad X(0) = X(1) = 0$$

$$\frac{1}{2} \frac{T'}{T} = \frac{X''}{X} = \lambda$$

$\lambda = 0, \lambda > 0$ produce zero solution

$$\lambda = -k^2 \quad X(x) = A \cos(kx) + B \sin(kx)$$

$$X(0) = A = 0$$

$$X(1) = B \sinh k = 0 \quad k = n\pi$$

$$\lambda = -n^2\pi^2 \\ T = C e^{-2n^2\pi^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-2n^2\pi^2 t} \sin(n\pi x)$$

$$u(x,0) = 1-x^2 = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$$A_n = \frac{4(1+(-1)^n)}{n^3\pi^3} + \frac{2}{n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4(1+(-1)^n)}{n^3\pi^3} + \frac{2}{n\pi} \right] e^{-2n^2\pi^2 t} \sin(n\pi x)$$

