Differential Equations Class Test 2024.

Time Allowed: **Fifty Minutes.**

Non Programmable Calculators may be used. All working must be shown.

Question One.

(i) Solve the initial value problem

$$y' = x^2y^2, \; y(0) = 1.$$

(ii) Find the solution of the second order ODE

$$x^2y'' + 2xy' + \left(4x^6 - 6\right) y = 0,$$

in terms of Bessel functions.

Question Two.

(i) Given that $y_1 = x$ is a solution of

$$x^2y'' + 4xy' - 4y = 0,$$

obtain a second linearly independent solution.

(ii) Use variation of parameters to solve the inhomogeneous equation

$$x^2y'' + 4xy' - 4y = x^4.$$

Question Three.

Use a power series of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to obtain the general solution of the ODE

$$y' = x^3y, \; y(0) = 1.$$

Useful Information Over the Page.

Solutions. DEs class Test 2024

Q1) (a) $y' = x^2 y^2$, $y(0)=1$. This is separable

$\frac{dy}{y^2} = x^2\,dx$. Integrate both sides

$$\int \frac{dy}{y^2} = \int x^2\,dx$$

$$-\frac{1}{y} = \frac{1}{3}x^3 + C$$

So $$y = \frac{-1}{\frac{1}{3}x^3 + C}$$

$y(0) = \frac{-1}{C} = 1$ $\therefore$ $C = -1$

$$\therefore \quad y(x) = \frac{-1}{\frac{1}{3}x^3 - 1} = \frac{1}{1 - \frac{1}{3}x^3}$$

$$= \frac{3}{3 - x^2}, \quad x^2 \neq 3.$$

(b) $x^2 y'' + 2xy' + (4x^6 - 6)y = 0$

$1 - 2s = 2$. $\quad s^2 - r^2\alpha^2 = -6$, $\quad a^2 r^2 x^{2r} = 4x^6$

$-1 = 2s$ $\qquad\qquad\qquad\qquad r = 3$

$s = -1/2$ $\qquad \frac{1}{4} - 9\alpha^2 = -6$ $\qquad 9a^2 = 4$

$9\alpha^2 = \frac{1}{4} + 6$ $\qquad a^2 = \frac{4}{9}$

$9\alpha^2 = \frac{25}{4}$ $\qquad a = 2/3$

$\alpha^2 = \frac{25}{36}$

$\alpha = \pm\frac{5}{6}$.

$$y = C_1 x^{-1/2} J_{\frac{5}{6}}\left(\tfrac{2}{3}x^3\right) + C_2 x^{-\frac{1}{2}} J_{-\frac{5}{6}}\left(\tfrac{2}{3}x^3\right)$$

Q2 $\quad x^2y'' + 4xy' - 4y = 0 \Rightarrow y'' + \frac{4}{x}y' - \frac{4}{x^2}y = 0$

$y_1 = x$. $\quad p(x) = \frac{4}{x}$ $\quad \int p(x) = 4\ln x$.

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx = x\int \frac{e^{-\ln x^4}}{x^2}\,dx.$$

$$= x\int \frac{1}{x^6}\,dx = x\left(-\frac{1}{5}x^{-5}\right) = -\frac{1}{5}x^{-4}$$

Take $y_2 = x^{-4}$

(b) $y'' + \frac{4}{x}y' - 4y = x^2$. $R(x) = x^2$

$$W(y_1, y_2) = \begin{vmatrix} x & x^{-4} \\ 1 & -4x^{-5} \end{vmatrix} = -4x^{-4} - x^{-4} = -5x^{-4}.$$

$y_p = y_1 u + y_2 v$

$$u' = -\frac{y_2 R}{W} = \frac{-x^{-4}x^2}{-5x^{-4}} = \frac{1}{5}x^2$$

$$u = \frac{1}{15}x^3$$

$$v' = \frac{x\,x^2}{-5x^{-4}} = -\frac{1}{5}x^7$$

$$v = -\frac{1}{40}x^8.$$

So $y_p = x\left(\frac{1}{15}x^3\right) - \frac{1}{40}x^{-4}x^8$

$$= \left(\frac{1}{15} - \frac{1}{40}\right)x^4 = \frac{40-15}{15\times 40}x^4$$

$$= \frac{25}{600}x^4 = \frac{5}{120}x^4$$

$$= \frac{1}{24}x^4$$

$$\therefore\ y = C_1 x + C_2 x^{-4} + \frac{1}{24}x^4.$$

Q3 $y' = x^3 y$. $y(0) = 1$. $y = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+3}$$

$$a_1 + 2a_2 x + 3a_3 x^2 + \sum_{n=4}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+3}$$

$a_1 = a_2 = a_3 = 0$. Shift starting value of

$$\sum_{n=0}^{\infty} \left[(n+4) a_{n+4} x^{n+3} - a_n x^{n+3} \right] = 0$$

$$a_{n+4} = \frac{a_n}{n+4}, \quad n \geq 0$$

$n=0 \qquad a_4 = \frac{1}{4} a_0$

$\cancel{n=1 \qquad a_5 = \frac{1}{5} a_4 = \frac{1}{4\times 5} a_0}$

$\cancel{n=2 \qquad a_7 =}$

$n=1 \qquad a_5 = \frac{1}{5} a_1 = 0$

$n=2 \qquad a_6 = \frac{1}{6} a_2 = 0$

$n=3 \qquad a_7 = \frac{1}{7} a_3 = 0$

$n=4 \qquad a_8 = \frac{1}{8} a_4 = \frac{a_0}{4\times 8}$

$n=5 \qquad a_9 = \frac{1}{9} a_5 = 0$

$n=6 \qquad a_{10} = \frac{1}{10} a_6 = 0, \quad n=7 \to a_{11} = \frac{1}{11} a_7 = 0$

$$a_{12} = \frac{a_8}{12} = \frac{a_0}{4\times 8\times 12} = \frac{a_0}{4^3 (1\times 2\times 3)}$$

$a_{4n} = \dfrac{a_0}{4^n \, n!}$ all other terms are zero.

$y = a_0 \displaystyle\sum_{n=0}^{\infty} \frac{x^{4n}}{n! \, 4^n} = a_0 e^{\frac{x^4}{4}} \quad y(0) = 1 \Rightarrow a_0 = 1.$