Differential Equations Class Test. 2025 Autumn

(1) Given that $y_1 = x$ is a solution of the ODE
$$x^2y'' + 4xy' - 4y = 0,$$
construct a second linearly independent solution.

(2) Use variation of parameters to find the general solution of
$$x^2y'' + 4xy' - 4y = x^4.$$

(3) Find two power series solutions of the equation
$$y'' + 4y = 0.$$
Express the solution in terms of elementary functions.

Solutions. Class Test DEs. 2025

(1) $x^2y'' + 4xy' - 4y = 0$. $y_1 = x$ is a solution

We divide by x^2 to obtain

$$y'' + \frac{4}{x}y' - \frac{4}{x^2}y = 0. \quad \text{So } p(x) = \frac{4}{x}$$

$\int p(x)dx = 4\ln x$.

Thus $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2}dx = x\int \frac{e^{-4\ln x}}{x^2}dx$

$$= x\int \frac{x^{-4}}{x^2}dx = x\int x^{-6}dx = -\frac{1}{5}x.x^{-5}$$

$$= -\frac{1}{5}x^{-4}.$$

(2) $x^2y'' + 4xy' - 4y = x^4$. Divide by x^2 to get

$$y'' + \frac{4}{x}y' - \frac{4}{x^2}y = x^2. \quad R(x) = x^2.$$

Take $y_1 = x$, $y_2 = x^{-4}$. Then the Wronskian is

$$w(y_1, y_2) = \begin{vmatrix} x & x^{-4} \\ 1 & -4x^{-5} \end{vmatrix} = -5x^{-4}$$

So we construct $y_p = y_1 u + y_2 v$

$$u' = -\frac{y_2 R}{w} = \frac{-x^{-4}(x^2)}{-5x^{-4}} = \frac{1}{5}x^2.$$

Take $u = \frac{1}{15}x^3$.

$$v' = \frac{y_1 R}{w} = \frac{x.x^2}{-5x^{-4}} = -\frac{1}{5}x^7, \quad v = -\frac{1}{40}x^8$$

$$y_p = x\left(\frac{1}{15}x^3\right) - \frac{1}{40}x^8(x^{-4}) = \frac{1}{24}x^4.$$

$$y = y_p + c_1 x + c_2 x^{-4}$$

(3) $y'' + 4y = 0$ $y = \sum_{n=0}^{\infty} a_n x^n$

Thus $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 4 a_n x^n$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 4 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 4a_n\right] x^n = 0$$

Hence $a_{n+2} = \dfrac{-4a_n}{(n+2)(n+1)}$ $n \geq 0$

now, $n=0 \Rightarrow a_2 = \dfrac{-4a_0}{1 \times 2}$

$n=2 \Rightarrow a_4 = \dfrac{(-4)^2 a_0}{1 \times 2 \times 3 \times 4}$

$n=4 \Rightarrow a_6 = \dfrac{(-4)^3 a_0}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = \dfrac{(-2)^6 a_0}{6!}$

In general $a_{2n} = \dfrac{(-2)^{2n} a_0}{(2n)!}$

So $y_1 = a_0 \sum_{n=0}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!} = a_0 \cos(2x)$ is a solution

$n=1 \Rightarrow a_3 = \dfrac{-4a_1}{1 \times 2 \times 3}$, $n=3 \Rightarrow a_5 = \dfrac{(-4)^2 a_1}{1 \times 2 \times 3 \times 4 \times 5}$

$a_{2n+1} = \dfrac{(-1)^n}{2} \dfrac{2^{2n+1} a_1}{(2n+1)!}$

$y_2 \; \dfrac{a_1}{2} \sum_{n=0}^{\infty} (-1)^n \dfrac{(2x)^{2n+1}}{(2n+1)!} = \dfrac{a_1}{2} \sin(2x)$

Gen solution is $y = a_0 \cos(2x) + \dfrac{a_1}{2} \sin(2x)$