DE Exam 2024 Solutions

1 (i) $x^3y'' - 2xy' + 2y = 0$

or $y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 0$

$y_1 = x$ is a solution.

$p(x) = -\frac{2}{x^2}$ $\quad\quad \int p(x) = -2\int x^{-2}\,dx$

$= \frac{2}{x}.$

$e^{-\int p(x)dx} = e^{-\frac{2}{x}}.$

$y_2 = y_1 \int \frac{e^{-\frac{2}{x}}}{x^2}\,dx.$ $\quad$ put $-\frac{2}{x} = u$ $\quad$ (4)

$du = \frac{2}{x^2}dx \therefore \frac{dx}{x^2} = \frac{1}{2}du$

$y_2 = x\int \frac{1}{2}e^u\,du = \frac{1}{2}xe^u = \frac{1}{2}xe^{-\frac{2}{x}}.$

(ii) $y'' - 4y' + 4y = e^{2x}$ $\quad\quad y_1 = e^{2x},\ y_2 = xe^{2x}$

$R(x) = e^{2x}.$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$ (6)

$= e^{4x} + 2xe^{4x} - 2xe^{4x}$

$= e^{4x}.$

$u' = \frac{-y_2R}{W} = \frac{-xe^{2x}e^{2x}}{e^{4x}} = -x.$ $\quad u = -\frac{1}{2}x^2$

$v' = \frac{y_1R}{W} = \frac{e^{2x}e^{2x}}{e^{4x}} = 1.$ $\quad$ So $v = x$

$y_p = y_1u + y_2v = -\frac{1}{2}x^2e^{2x} + x^2e^{2x} = \frac{1}{2}x^2e^{2x}.$

Q(2) $y'' - 2xy' - y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

So
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} a_n x^n =$$

$$2a_2 - a_0 + \sum_{n=3}^{\infty} n(n-1)a_{n} x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} a_n x^n$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} (2n+1)a_n x^n$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - (2n+1)a_n\right] x^n = 0$$

$\therefore\ a_2 = \frac{1}{2} a_0 \qquad a_{n+2} = \frac{(2n+1)}{(n+2)(n+1)} a_n, \ n \geq 1.$

$n=2, \qquad a_4 = \frac{5}{4 \times 3} a_2 = \frac{5}{2 \times 3 \times 4} a_0$

$n=4 \qquad a_6 = \frac{9 \times 5}{6 \times 5 \times 4 \times 3 \times 2} a_0$

$n=6 \qquad a_8 = \frac{5 \times 9 \times 13}{8!} a_0$

For numerator last term is $an+b$
$n=3 \Rightarrow 3a+b=9$
$n=4 \Rightarrow 4a+b=13$
So $a=4$
$b=-3$.

$$a_{2n} = \frac{5 \times \cdots \times (4n-3)}{(2n)!} a_0$$

$$y_1 = a_0\left(1 + \sum_{n=1}^{\infty} \frac{5 \times \cdots \times (4n-3)}{(2n)!} x^{2n}\right)$$

$n=1, \quad a_3 = \frac{3}{3\times 2} a_1,$

$n=3 \quad a_5 = \frac{7}{5\times 4} a_3 = \frac{3\times 7}{5!} a_1$

$n=5 \quad a_7 = \frac{3\times 7\times 11}{7!} a_1 \quad$ etc

$$a_{2n+1} = \frac{3\times \cdots \times (4n-1)}{(2n+1)!} a_1 \qquad n \geq 1$$

$$y_2 = a_1\left(x + \sum_{n=1}^{\infty} \frac{3\times \cdots \times (4n-1)}{(2n+1)!} x^{2n+1}\right)$$

Q3 $\quad xy'' + 2y' - y = 0 \qquad y = \sum_{n=0}^{\infty} a_n x^{n+s}$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + 2\sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= \left(s(s-1) + 2s\right) a_0 x^{s-1} + \sum_{n=1}^{\infty} \left[(n+s)(n+s-1) + 2(n+s)\right] a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= (s^2+s) a_0 x^{s-1} + \sum_{n=1}^{\infty} \left(\left[(n+s)(n+s-1) + 2(n+s)\right] a_n - a_{n-1}\right) x^{n+s-1}$$

$= 0$

$s^2 + s = 0 \qquad s = 0$ or -1

$$a_n = \frac{a_{n-1}}{(n+s)(n+s+1)}$$

If $s=0 \qquad a_n = \frac{a_{n-1}}{n(n+1)}$

$$a_n = \frac{a_0}{n!(n+1)!}$$

So $y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!}$,

(ii) $x^2y'' + 2xy' - xy = 0$

$1-2s = 2$, $s^2 - r^2\alpha^2 = 0$

$s = -\frac{1}{2}$ $a^2r^2x^{2r} = -x$. $r = \frac{1}{2}$

$a = i$.

$\frac{1}{4} - \frac{1}{4}\alpha^2 = 0$ $\alpha = \pm 1$.

Solution is $y = c_1 x^{-\frac{1}{2}} I_1\left(\frac{x^{\frac{1}{2}}}{2}\right) + c_2 x^{-\frac{1}{2}} K_1\left(\frac{x^{\frac{1}{2}}}{2}\right)$.

Q4. (i) $y'' + 5y' + 6y = e^{-x}$, $y(0) = 0$, $y'(0) = 0$.

$\mathcal{L}(y'' + 5y' + 6y) = \mathcal{L}(e^{-x})$ $\mathcal{L}(y) = Y$

$(s^2 + 5s + 6)Y(s) = \frac{1}{s+1}$.

$Y(s) = \frac{1}{(s+1)(s+2)(s+3)}$

$= \frac{1}{2}\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2(s+3)}$

$y(x) = \frac{1}{2}e^{-x} - e^{-2x} + \frac{1}{2}e^{-3x}$.

(ii) $F(s) = \frac{s}{(s^2+4)\sqrt{4+s^2}}$

$f(t) = \int_0^t \cos(2(t-u))\, J_0(2u)\, du$

Q5. (i) $b_n = 2\int_0^1 (1-x^2)\sin(n\pi x)\,dx$

$$= \frac{4 + 2n^2\pi^2 - 4(-1)^n}{n^3\pi^3}$$

$$= \frac{4(1-(-1)^n)}{n^3\pi^3} + \frac{2}{n\pi}$$

(ii) $\frac{1}{2}u_t = u_{xx} \quad 0 < x \leq 1$, $u(x,0) = f(x)$

$u = XT$, $X(0) = X(1) = 0$

$$\frac{1}{2}\frac{T'}{T} = \frac{X''}{X} = \lambda$$

$\lambda = 0$, $\lambda > 0$ produce zero solution

$\lambda = -k^2 \quad X(x) = A\cos(kx) + B\sin(kx)$

$X(0) = A = 0$

$X(1) = B\sin k = 0 \quad k = n\pi$

$\lambda = -n^2\pi^2$

$T = Ce^{-2n^2\pi^2 t}$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-2n^2\pi^2 t}\sin(n\pi x)$$

$$u(x,0) = 1 - x^2 = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$$A_n = \frac{4(1-(-1)^n)}{n^3\pi^3} + \frac{2}{n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{4(1-(-1)^n)}{n^3\pi^3} + \frac{2}{n\pi}\right]e^{-2n^2\pi^2 t}\sin(n\pi x)$$