DE's Class Test Solutions
2022

Question One

$$x^2y'' + 4xy' - 4y = x^6$$

$y = x^r$

$$x^2 r(r-1)x^{r-2} + 4x\,r x^{r-1} - 4x = 0$$

$$r^2 + 3r - 4 = 0$$

$r = 1, -4.$

$y_1 = x, \quad y_2 = x^{-4}.$

Wronskian $W = \begin{vmatrix} x & x^{-4} \\ 1 & -4x^{-5} \end{vmatrix} = -4x^{-4} - x^{-4}$
$= -5x^{-4}$

So we have $y'' + \frac{4}{x}y' - \frac{4}{x^2}y = x^4$

$R(x) = x^4$

$$y_p = uy_1 + vy_2$$

$$u' = -\frac{y_2 R}{W} = \frac{-x^{-4}x^4}{-5x^{-4}} = \frac{1}{5}x^4$$

$$u = \frac{1}{25}x^5$$

$$v' = \frac{y_1 R}{W} = \frac{x x^4}{-5x^{-4}} = \frac{-1}{5}x^9$$

$$v = \frac{-1}{50}x^{10}.$$

$$y_p = x\frac{1}{25}x^5 - \frac{1}{50}x^{-4}x^{10} = \frac{1}{50}x^6$$

So $y = c_1x + c_2x^{-4} + \frac{1}{50}x^6$

Question Two

$$y'' + 2xy' - 4y = 0 \qquad y = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n - 4\sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 - 4a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty}(2n-4)a_n x^n$$

$$= 2a_2 - 4a_0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$+ \sum_{n=1}^{\infty} 2(n-2) a_n x^n = 0$$

$$a_2 = 2a_0 , \quad \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} + 2(n-2) a_n \right) x^n = 0$$

$$a_{n+2} = \frac{2(2-n) a_n}{(n+2)(n+1)}$$

$n=0 \qquad a_2 = 2a_0$

$n=2 \qquad a_4 = \dfrac{2(2-2)a_2}{4 \cdot 3} = 0$

So $n=4 \qquad a_6 = \dfrac{2(2-4)a_4}{6 \cdot 5} = 0$

all even terms after a_2 are zero

So $y = a_0(1+2x^2)$ is a solution.

For the odd

$n=1 \qquad a_3 = \dfrac{2(2-1)\, a_1}{3 \cdot 2} = \dfrac{2\, a_1}{2 \cdot 3}$

$n=3 \qquad a_5 = \dfrac{2(2-3)}{5 \cdot 4} = \dfrac{2^2 (1 \times -1)\, a_1}{2 \times 3 \times 4 \times 5}$

$n=5 \qquad a_7 = \dfrac{2(2-5)}{7 \cdot 6} = \dfrac{2^3}{7!}(1 \times (-1) \times -3)\, a_1$

$n=7 \qquad a_9 = \dfrac{2(2-7)}{9 \cdot 7} = (-1)^3 \dfrac{2^4 (1 \times 1 \times 3 \times 5)}{9!} a_1$

$$a_{2n+1} = (-1)^n a_1 \frac{2^n}{(2n+1)!} (1 \times 1 \times 3 \times \dots \times (2n-1)) \quad n \geq 1$$

$$\therefore y = a_0(1+2x^2) + a_1 \left(x + \sum_{n=1}^{\infty} (-1)^n \frac{2^n (1 \times 3 \times \dots (2n-1))}{(2n+1)!} x^{2n+1} \right)$$

Question Three Use LT to solve

$$y'' + 4y = \sin t$$
$$y(0) = 0, \quad y'(0) = 1$$

Take LT,

$$(s^2+4)Y(s) - y'(0) = \frac{1}{s^2+1} \quad \text{as } y(0) = 0.$$

So $$Y(s) = \frac{1}{s^2+4} + \frac{1}{(s^2+4)(s^2+1)}$$

Now $$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$\therefore \; Y(s) = \frac{1}{3}\frac{1}{s^2+1} + \frac{2}{3}\frac{1}{s^2+4}$$
$$= \frac{1}{3}\frac{1}{s^2+1} + \frac{1}{3}\frac{2}{s^2+4}$$

$$y(t) = \frac{1}{3}(\sin t + \sin(2t))$$