Autumn 2023 – **Main Exam**

UNIVERSITY OF TECHNOLOGY, SYDNEY

STUDENT NUMBER: | | | | | | | | |

SURNAME:
(FAMILY NAME) ....

OTHER NAMES: ....

This paper and all materials issued must be returned at the end of the examination.
They are not to be removed from the exam centre.

37335 Differential Equations

Time Allowed: 2 hours and 10 mins

Includes 10 minutes of reading time.

Reading time is for reading only. You are not permitted to write, calculate or mark your paper in any way during reading time.

This is an Open Book exam

Unauthorised materials as specified in the examination conditions are not allowed.

Permitted materials for this exam:

- Calculators (non-programmable only)

Materials provided for this exam:

- This examination paper
- One Formula sheet at end of exam paper.

Students please note:

- Answer All Questions.

Examination Conditions:

It is your responsibility to fill out and complete your details in the space provided on all the examination material provided to you. Use the time before your examination to do so as you will not be allowed any extra time once the exam has ended.

You are **not** permitted to have on your desk or on your person any unauthorised material. This includes but not limited to:

- Mobile phones
- Smart watches and bands
- Electronic devices
- Draft paper (unless provided)
- Textbooks (unless specified)
- Notes (unless specified)

You are **not** permitted to obtain assistance by improper means or ask for help from or give help to any other person.\

If you wish to **leave and be re-admitted** (including to use the toilet), you have to wait until **90 mins** has elapsed.

If you wish to **leave the exam room permanently**, you have to wait until **60 mins** has elapsed.

You are not permitted to leave your seat (including to use the toilet) during the final 15 mins.

During the examination **you must first seek permission** (by raising your hand) from a supervisor before:

- Leaving early
- Using the toilet
- Accessing your bag

Disciplinary action will be taken against you if you infringe university rules.

All Working Must Be Shown For Every Question.

Question 1 $(4 + 6 = 10$ marks).

(i) The ordinary differential equation

$$x^3y'' - 4xy' + 4y = 0,$$

has a solution $y_1(x) = x$. Use this to find a second linearly independent solution y_2.

(ii) Use variation of parameters to find the general solution of the ODE

$$y'' - 4y' + 4y = e^x.$$

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Question 2 (10 marks).

(a) Obtain the general solution of the ODE

$$y'' + 4xy' + y = 0,$$

by letting $y = \sum_{n=0}^{\infty} a_n x^n$.

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Question 3 (8+2=10 marks).

(i) Look for solutions of the equation

$$x^2y'' + 3xy' + \left(x + \frac{8}{9}\right)y = 0,$$

of the form $y = \sum_{n=0}^{\infty} a_n x^{n+s}$.

Show that the exponents are $s_1 = -4/3, s_2 = -2/3$ and that a_n satisfies

$$a_n = \frac{-9a_{n-1}}{9(n+s)(n+s+2)+8}.$$

Hence obtain the general solution of the ODE.

(ii) Identify the solution of the equation in (i) in terms of Bessel functions.

Question 4 (7+3=10 marks)

(i) Use the Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = e^{-3x}, \tag{0.1}$$

$y(0) = 0, y'(0) = 0$. Note that we can write

$$\frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3},$$

for certain constants A, B, C.

(ii) Obtain the inverse Laplace transform of

$$F(s) = \frac{1}{(s^2+1)\sqrt{1+s^2}},$$

as an integral. The transforms you need are in the table at the back of the exam.

Question 5 (3+5+2=10 marks).

(i) Calculate the Fourier sine series for the function

$$f(x) = 1 - x, \quad 0 \le x \le 1.$$

(ii) Consider the following initial and boundary value problem for the heat equation.

$$\begin{aligned} \frac{1}{4}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= 1 - x. \end{aligned}$$

By looking for solutions of the heat equation of the form $u(x,t) = X(x)T(t)$ show that the functions X and T must satisfy the problems

$$X''(x) = \lambda X(x), \quad X(0) = X(1) = 0,$$

and

$$T'(t) = 4\lambda T(t)$$

for some constant λ. Explain how the possible values of λ are obtained. Hence obtain an expression for X and T.

(iii) Use your answer for part (i) and part (ii) to write down the solution of the given initial and boundary value problem for the heat equation.

Table of integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

$$\int \frac{du}{u} = \log|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\begin{aligned}\int \frac{du}{\sqrt{a^2+u^2}} &= \sinh^{-1}\frac{u}{a} + C\\ &= \log\left(u+\sqrt{a^2+u^2}\right) + C\end{aligned}$$

$$\begin{aligned}\int \frac{du}{1-u^2} &= \tanh^{-1} u + C_1\\ &= \frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + C_2\end{aligned}$$

$$\int \cosh u\,du = \sinh u + C$$

$$\int \sinh u\,du = \cosh u + C$$

$$\int \tanh u\,du = \log\cosh u + C$$

$$\int u\,dv = uv - \int v\,du$$

Table of Laplace transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}(e^{-at}) = \frac{1}{s+a}$$
$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$
$$\mathcal{L}(J_0(t)) = \frac{1}{\sqrt{1+s^2}}.$$

Variation of parameters

Given that y_1 and y_2 are solutions of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

we seek a particular solution of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

by looking for solutions of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. The functions u and v must satisfy

$$u'y_1 + v'y_2 = 0,$$
$$u'y_1' + v'y_2' = f(x).$$

Bessel Functions

Bessel's differential equation $t^2u'' + tu' + (t^2 - \alpha^2)u = 0$ may be transformed into the equation

$$x^2y'' + (1-2s)xy' + ((s^2 - r^2\alpha^2) + a^2r^2x^{2r})y = 0$$

under the change of variables $t = ax^r$ and $y(x) = x^s u(t)$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$