1. (i) $\int \underbrace{x}_{u} \underbrace{\sin x\, dx}_{dv} = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C$

$u = x \rightarrow du = dx$

$dv = \sin x\, dx \rightarrow v = -\cos x$

INTEGRATION BY PARTS

$\int u\,dv = uv - \int v\,du$

(ii) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u} = -\log u + C = -\log(\cos x) + C$

Let $u = \cos x$. Then $du = -\sin x\, dx$

(iii) $\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{du}{u} = \log|u| + C = \log|\sec x + \tan x| + C$

Multiply and divide by $\sec x + \tan x$

set $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)\, dx$

(iv) $\int \frac{x\, dx}{(x+1)(x+2)} = -\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2} = -\log|x+1| + 2\log|x+2| + C$

$\frac{x}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{2}{x+2}$

partial fraction decomposition

2 (i) $\frac{dy}{dx} + 4y = 0, \quad y(0) = 2$

$\frac{dy}{dx} = -4y$

$\frac{dy}{y} = -4\, dx$

Integrate both sides:

$\int \frac{dy}{y} = -4\int dx$

$\log y = -4x + C$

Take exponentials on both sides:

$y = e^{-4x+C} = Ke^{-4x}$, where $K = e^C$

Now $y(0) = 2$ gives

$2 = Ke^0 = K$. $\Rightarrow K = 2$ and so $\boxed{y(x) = 2e^{-4x}}$

(ii) $y' + 4y = 6x$, $y(0) = 5$

- First we solve the homogeneous problem $y' + 4y = 0$

  From the previous exercise we know that a solution to the homogeneous problem is given by

  $$y_h = Ke^{-4x}$$

- Now we look for a particular solution of the form $y_p = Ax + B$

  $y_p' = A$

  substitute y_p and y_p' into the equation:

  $$A + 4(Ax + B) = 6x \Rightarrow (A + 4B) + 4Ax = 6x \Rightarrow \begin{cases} 4A = 6 \Rightarrow A = \frac{3}{2} \\ A + 4B = 0 \Rightarrow \frac{3}{2} + 4B = 0 \Rightarrow B = -\frac{3}{8} \end{cases}$$

  So $y_p = \frac{3}{2}x - \frac{3}{8}$

- The general solution is $y(x) = y_h + y_p = Ke^{-4x} + \frac{3}{2}x - \frac{3}{8}$

  Now the initial condition $y(0) = 5$ gives

  $$5 = Ke^0 + 0 - \frac{3}{8} \Leftrightarrow 5 = K - \frac{3}{8} \Leftrightarrow K = 5 + \frac{3}{8} = \frac{48}{8}$$

  So $\boxed{y(x) = \frac{43}{8}e^{-4x} + \frac{3}{2}x - \frac{3}{8}}$

(iii) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$, $y(0) = 1$, $y'(0) = 2$

- The characteristic/auxiliary equation is

  $$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = \frac{-3 \pm \sqrt{9 - 8}}{2} = \frac{-3 \pm 1}{2} \begin{cases} -\frac{4}{2} = -2 \\ -\frac{2}{2} = -1 \end{cases}$$

  So the general solution is

  $$y(x) = C_1 e^{-2t} + C_2 e^{-t}$$

- To determine C_1 and C_2 we compute y' and we use the initial conditions $y(0) = 1$ and $y'(0) = 2$:

  $$y'(x) = -2C_1 e^{-2t} - C_2 e^{-t}$$

  $$y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2 = 1 \Rightarrow C_1 = 1 - C_2$$

  $$y'(0) = -2C_1 e^0 - C_2 e^0 = -2C_1 - C_2 = 2$$

  $$-2(1 - C_2) - C_2 = 2$$

  $$-2 + 2C_2 - C_2 = 2 \Rightarrow \boxed{C_2 = 4} \Rightarrow C_1 = 1 - 4 = -3$$

  $$\boxed{C_1 = -3}$$

So $y(x) = -3e^{-2t} + 4e^{-t}$

(iv) $\frac{d^2y}{dx^2} + 4y = \sin x$

- First, solve the homogeneous problem $y'' + 4y = 0$, which has characteristic equation $\lambda^2 + 4 = 0$

$$\Leftrightarrow \lambda^2 = -4 \Leftrightarrow \lambda = \pm\sqrt{-4} \Leftrightarrow \lambda = \pm 2i$$

So the solution to the homogeneous problem is $y_h = c_1\cos(2x) + c_2\sin(2x)$

- Now we look for a particular solution of the form

$y_p = A\sin x + B\cos x \Rightarrow y_p' = A\cos x - B\sin x$, $y_p'' = -A\sin x - B\cos x$

Substitute y_p and y_p'' into the equation to find A and B:

$$-A\sin x - B\cos x + 4(A\sin x + B\cos x) = \sin x$$

$$3A\sin x + 3B\cos x = \sin x \Leftrightarrow \begin{cases} 3A = 1 \Leftrightarrow A = \frac{1}{3} \\ 3B = 0 \Leftrightarrow B = 0 \end{cases}$$

So $y_p = \frac{1}{3}\sin x$

- The general solution is $y(x) = y_h + y_p = \boxed{c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin x}$

(v) $y'' + 5y' + 25y = 0$, $y(0) = 1$, $y'(0) = 2$

- The characteristic equation is

$$\lambda^2 + 5\lambda + 25 = 0$$

$$\lambda = \frac{-5 \pm \sqrt{25 - 100}}{2} = \frac{-5 \pm \sqrt{-75}}{2} = \frac{-5 \pm 5\sqrt{3}i}{2} = -\frac{5}{2} \pm \frac{5\sqrt{3}}{2}i$$

So the general solution is

$$y(x) = e^{-\frac{5}{2}t}\left(c_1\cos\left(\frac{5\sqrt{3}}{2}t\right) + c_2\sin\left(\frac{5\sqrt{3}}{2}t\right)\right)$$

- To determine c_1 and c_2, compute $y'(x)$ and use the initial conditions $y(0) = 1$, $y'(0) = 2$:

$$y'(x) = e^{-\frac{5}{2}t}\left[-\frac{5}{2}c_1\cos\left(\frac{5\sqrt{3}}{2}t\right) + \left(-\frac{5}{2}c_2\right)\sin\left(\frac{5\sqrt{3}}{2}t\right) + \left(-\frac{5\sqrt{3}}{2}c_1\right)\sin\left(\frac{5\sqrt{3}}{2}t\right) + \frac{5\sqrt{3}}{2}c_2\cos\left(\frac{5\sqrt{3}}{2}t\right)\right]$$

$$= e^{-\frac{5}{2}t}\left[\left(\frac{5\sqrt{3}}{2}c_2 - \frac{5}{2}c_1\right)\cos\left(\frac{5\sqrt{3}}{2}t\right) + \left(-\frac{5\sqrt{3}}{2}c_1 - \frac{5}{2}c_2\right)\sin\left(\frac{5\sqrt{3}}{2}t\right)\right]$$

Now

$$y(0) = e^0 \left[c_1 \cos 0 + \cancel{c_2 \sin 0} \right] = \boxed{c_1 = 1}$$

$$y'(0) = e^0 \left[\left(\frac{5\sqrt{3}}{2} c_2 - \frac{5}{2} c_1 \right) \cos 0 + \cancel{\left(-\frac{5\sqrt{3}}{2} c_1 - \frac{5}{2} c_2 \right) \sin 0} \right] = \frac{5\sqrt{3}}{2} c_2 - \frac{5}{2} c_1 = 2$$

$$\Leftrightarrow \frac{5\sqrt{3}}{2} c_2 - \frac{5}{2} = 2 \Leftrightarrow \frac{5\sqrt{3}}{2} c_2 = \frac{9}{2} \Leftrightarrow \boxed{c_2 = \frac{3\sqrt{3}}{5}}$$

so $y(x) = e^{-\frac{5}{2}t} \left[\cos\left(\frac{5\sqrt{3}}{2} t \right) + \frac{3\sqrt{3}}{5} \sin\left(\frac{5\sqrt{3}}{2} t \right) \right]$