

Separation of Variables. Why does it work

Suppose we have $\frac{dy}{dx} = g(y)f(x)$,

Then we need to show that

$$\int \frac{1}{g(y)} dy = \int f(x) dx, \quad y \text{ is a function of } x.$$

Let $y_0 = y(x_0)$. Then

$$\int_{y_0}^y \frac{dy'}{g(y')} = \int_{x_0}^x f(u) du \quad (*)$$

Differentiate both sides wrt. x

$$\frac{d}{dx} \int_{x_0}^x f(u) du = f(x) \text{ by the Fundamental Theorem of Calculus.}$$

Next we use the chain rule

$$\begin{aligned} \frac{d}{dx} \int_{y_0}^y \frac{dy'}{g(y')} &= \frac{d}{dy} \left(\int_{y_0}^y \frac{dy'}{g(y')} \right) \frac{dy}{dx} \\ &= \frac{1}{g(y)} \frac{dy}{dx} \end{aligned}$$

Hence $\frac{dy}{dx} = g(y)f(x)$. So (*) gives the solution of the equation.