Differential Equations. Workshop 1
Revision

A differential equation is a relationship between a function and its derivatives. An example is

$$\frac{dy}{dx} = x.$$

This is one of the simplest possible equations.

The problem? Find y.

In this case the answer is easy. We integrate both sides with respect to x.

$$\int \frac{dy}{dx}\, dx = \int x\, dx$$

Since integration undoes differentiation (and vice versa)

$$\int \frac{dy}{dx}\, dx = y$$

$$\int x\, dx = \frac{1}{2}x^2 + C, \quad C \text{ a constant}$$

We have

$$y = \frac{1}{2}x^2 + C.$$

Notice that for each C, we get a different y. So the equation has infinitely many solutions.

We can get a unique solution by adding a condition on y.

Problem Find y such that

$$\frac{dy}{dx} = x$$

and $y(0) = 1$.

Solution

As before, $y = \frac{1}{2}x^2 + C$.

However we also have $y(0) = 1$.

So $y(0) = \frac{1}{2}(0)^2 + C = C = 1$.

Thus $C = 1$ and the solution is

$$y = \frac{1}{2}x^2 + 1.$$

If we require a condition such as $y(0) = 1$, $y'(0) = 2$ [y' is the derivative of y]

Then we say that we have an initial value problem

A typical example looks like this.

$$y'' + 3y' + 2y = 0$$

$y(0) = 0$, $y'(0) = 1$ — The initial conditions.

This is a second order equation, because the highest order derivative is y'' - the second derivative.

The equation is also <u>linear</u>

A <u>linear differential equation</u> has the following form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x)y = 0$$

The functions $a_0(x), \ldots, a_n(x)$ are called the <u>coefficient functions</u>. We usually require them to be continuous and more often than not, <u>smooth</u>.

By smooth we mean that they can be <u>differentiated</u> as many times as needed.

In most cases we consider they can be differentiated infinitely often. We will come back to this.

<u>Notation</u> we often write

$$Ly = a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y$$

So we can think of L as

$$L = a_n(x)\frac{d^n}{dx^n} + \cdots + a_0(x)$$

L is a differential operator. This is very convenienient

It is <u>easier</u> to write $Ly = f(x)$ than

$$a_n(x)\frac{d^n y}{dx^n} + \dots + a_0(x)y = f(x)$$

The equation $Ly = 0$ is said to be <u>homogeneous</u>.

The equation $Ly = f(x)$ is said to be inhomogeneous, provided that $f(x) \neq 0$

One important question which we will answer is the following:

If we can solve $Ly = 0$ can we solve $Ly = f$, $f \neq 0$.

Under mild assumptions the answer is <u>yes</u>. This will be one of our major results.

Notice that because differentiation is linear, ie $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

then

$$L(y_1 + y_2) = Ly_1 + Ly_2$$

and $L(ay_1) = aLy_1$, when a is a constant.

So we say $Ly = 0$ is a <u>linear</u>

differential equation. Our focus will be on linear equations in the case $n=2$, which is the most important.

Notice that if $Ly_1 = 0$ and $Ly_2 = 0$, then $L(y_1+y_2) = Ly_1 + Ly_2 = 0$.
So if y_1, y_2 are solutions of $Ly=0$ then so is $y = y_1 + y_2$.

Adding solutions of linear equations together produces new solutions

This is a simple fact, but it is enomously useful and important. Many techniques for the solution of DEs (Differential Equations) only work for non linear equations.

<u>Non linear equations</u> We will solve some first order non linear equatons but mostly these are too difficult for us. In fact they are mostly too difficult for anybody.

The equation $y' = y^2$

is non linear, though it is not hard

It is non linear because of the y^2 term.

Suppose y_1, y_2 are solutions

Then $(y_1+y_2)' = y_1' + y_2' = y_1^2 + y_2^2$
$$\neq (y_1+y_2)^2$$

So y_1+y_2 is <u>not</u> a solution

A harder example is
$$y'' + y^2 = 0.$$

This is non linear (Check this) and is much harder.

<u>Moral</u> Non linear equations are usually very difficult.

Where do DEs come from, i.e. why should we care?

Orignally they came from physics. Take Newton's second law
$$F = ma$$
a is acceleration. Now if the position of a particle at time t is, x, then
$$\text{acceleration} = \frac{d^2x}{dt^2}.$$

Now suppose that $F = -kx$, $k > 0$

Then we have an equation for x

$$\frac{d^2x}{dt^2} = -kx.$$

You may recognise this as the equation for simple harmonic motion

There are countless examples.
The equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

is called Bessel's equation. It arises in, for example the study of wave motion. We will learn how to solve this equation and many others. To do so we will need new functions.

Integration

To solve differential equations we must be able to integrate. The most important rules are

(1) Integration by parts.

$$\int fg' = fg - \int f'g.$$

(2) Integration by substitution

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $u = g(x)$

(3) Partial fractions This is for rational functions $\frac{p(x)}{q(x)}$ where p, q are polynomials.

We will see examples later.

Basic Differential Equations

(1) Separable equations. These have the form.

$$\frac{dy}{dx} = f(x)\,g(y)$$

We can write this as

$$\frac{dy}{g(y)} = f(x)\,dx$$

All y's here All x's here

Integrate both sides

Example

$$\frac{dy}{dx} = ky, \quad y(0) = y_0$$

$$\frac{dy}{y} = k\,dx$$

So $\displaystyle\int \frac{dy}{y} = \int k\,dx$

We immediately see that we have

to be able to integrate. Here it is easy

$$\int \frac{dy}{y} = \ln y$$

$$\int k\,dx = kx + C$$

So $\ln y = kx + C$

or $y = e^{kx+C} = Ae^{kx}$

$A = e^{C}$.

Now $y(0) = Ae^{0} = A = y_0$

Thus $y = y_0 e^{kx}$

<u>Example 2</u> $y' = xy$, $y(0) = 1$

$$\frac{dy}{y} = x\,dx$$

or $\int \frac{dy}{y} = \int x\,dx$

$$\ln y = \frac{x^2}{2} + C$$

$$\therefore y = e^{c} e^{x^2/2} = Ae^{\frac{x^2}{2}}$$

$$y(0) = A = 1$$

$$\therefore y = e^{x^2/2}$$

<u>Example 3</u> $\frac{dy}{dx} = y^2$, $y(0) = 1$

$$\frac{dy}{y^2} = dx$$

$$\therefore \int y^{-2}\,dy = \int dx$$

or $$-\frac{1}{y} = x + C.$$

Hence $$y = \frac{-1}{x+C}.$$

$$y(0) = -\frac{1}{C} = 1 \quad \therefore \quad C = -1$$

And $$y = \frac{-1}{x-1} = \frac{1}{1-x}.$$

Notice that we only use 1 constant of integration. Why?

$$\int \frac{dy}{y^2} = -\frac{1}{y} + C_1$$

$$\int dx = x + C_2$$

So we have

$$-\frac{1}{y} + C_1 = x + C_2$$

or $$-\frac{1}{y} = x + \underbrace{C_2 - C_1}_{C}$$

So we just combine the two constants.

First order linear equations

These have the form

$$p(x)\frac{dy}{dx} + q(x)y = f(x) \qquad p(x) \neq 0$$

We can write this as

$$\frac{dy}{dx} + \frac{q(x)}{p(x)}y = \frac{f(x)}{p(x)}$$

or $$\frac{dy}{dx} + a(x)y = b(x) \qquad (*)$$

Now $$\frac{d}{dx}\left(e^{\int a(x)}y\right) = e^{\int a(x)}y' + a(x)e^{\int a(x)}y$$

Thus we multiply $(*)$ by $e^{\int a(x)}$

Then $$\underbrace{e^{\int a(x)dx}\frac{dy}{dx} + a(x)e^{\int a(x)y}y}_{} = b(x)e^{\int a(x)dx}$$

$$\frac{d}{dx}\left(e^{\int a(x)dx}y\right) = b(x)e^{\int a(x)dx}$$

Gives us the solution

Example $\frac{dy}{dx} + y = x$

$$e^{\int 1dx} = e^{x}$$

So $\underbrace{e^x y' + e^x y}$ $= xe^x$

$$\frac{d}{dx}(e^x y) = xe^x$$

$\therefore$ $e^x y = \int xe^x \, dx$

$$= xe^x - \int e^x \, dx$$

$$= xe^x - e^x + C$$

$$= e^x(x-1) + C$$

or $y = x - 1 + Ce^{-x}$.

There are more examples in the notes

<u>Second order constant coefficient</u>

These have the form

$$a_1 \frac{d^2y}{dx} + a_2 \frac{dy}{dx} + a_3 y = 0$$

We will consider non zero right hand sides later

Again we can write this as

$$y'' + p y' + q y = 0$$

These are the easiest to solve
we try solutions of the form
$y = e^{\lambda x}$.

Then $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

Hence $(\lambda^2 + p\lambda + q)e^{\lambda x} = 0$

or $\lambda^2 + p\lambda + q = 0$

<u>Case 1</u> $(\lambda - m_1)(\lambda - m_2) = 0$

(Two real roots)

Solution $y = Ae^{m_1 x} + Be^{m_2 x}$

<u>Case 2</u> A single root $\lambda = m$

solution $y = (Ax + B)e^{mx}$

<u>Case 3</u> $\lambda = \alpha + i\beta,\ \alpha - i\beta$

Solution $y = \cancel{B}e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$