Differential Equations

We start with first order equations. The basic situation comes from Newton's law

$$F = ma$$

Example A skydiver jumps from a plane. At a certain time the parachute is opened and the descent slows.

$F_g = -mg$ m mass of diver

$g = 9.81 \ m/s^2$ – acceleration due to gravity

$$a = \frac{dv}{dt}$$

$$m\frac{dv}{dt} = -mg$$

When ripchord is pulled there is a new force $F_d = kv^2$ (F_d = drag force)

we have $m\frac{dv}{dt} = -mg + kv^2$

Say chute opens at $t = T_1$

$$v(T_1) = -gT_1$$

So how do we find v? Obviously we solve the equation.

First order Separable Equations

Let $\frac{dy}{dx} = a(x)g(y)$. We write

$$\frac{dy}{g(y)} = a(x)\,dx$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int a(x)\,dx$$

Example 2 P is the number of bacteria in a petri dish.

Logistic growth Here (p17)

$$\frac{dP}{dt} = k(M-P)P, \qquad P(0) = P_0$$

To solve we write

$$\frac{dP}{(M-P)P} = k\,dt$$

or
$$\int \frac{dP}{(M-P)P} = \int k\,dt$$

$$\int \frac{dP}{(M-P)P} = \int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP$$

$$= \frac{1}{M}\left(\ln P - \ln(M-P)\right) + C$$

$$= \frac{1}{M} \ln\left(\frac{P}{M-P}\right) + C$$

So
$$\frac{1}{M} \ln\left(\frac{P}{M-P}\right) + C = \int k\,dt = kt + C_1$$

$$\ln\left(\frac{P}{M-P}\right) = Mkt + D$$

Take the log of both sides

$$\frac{P}{M-P} = Ae^{Mkt} \qquad A = e^D$$

$$\therefore \quad P = (M-P)Ae^{Mkt}$$

$$P(1 + Ae^{Mkt}) = MAe^{Mkt}$$

or
$$P(t) = \frac{MAe^{Mkt}}{1 + Ae^{Mkt}}$$

To find A, we let $P(0) = P_0$

So $\dfrac{MA}{1+A} = P_0$ Solve for A.

This gives
$$A = \frac{P_0}{M-P_0} \quad \text{(Calc. in notes)}$$

<u>Example 3</u> $\dfrac{dy}{dx} = (x^2+3)(y^2+y^3)$

So $\dfrac{dy}{y^2+y^3} = (x^2+3)\,dx$

$$\int \frac{dy}{y^2+y^3} = \int\left(\frac{1}{y^2} + \frac{1}{y+1} - \frac{1}{y}\right) dy$$

$$= -\frac{1}{y} + \ln(y+1) - \ln y = \int (x^2+3)\,dx$$

$$= \frac{x^3}{3} + 3x + C$$

$$= -\frac{1}{y} + \ln\left(\frac{y+1}{y}\right) = \frac{x^3}{3} + 3x + C.$$

Take exponential of both sides
$$\left(\frac{y+1}{y}\right)e^{-1/y} = A e^{\frac{x^3}{3}+3x}, \quad A = e^C$$

This cannot be solved for y. So we have an implicit solution. This is very common. In fact it is typical.

Return to Problem 1
$$m\frac{dv}{dt} = -mg + kv^2, \quad v(T_1) = -gT_1$$

So $\displaystyle\int \frac{dv}{-g + \frac{k}{m}v^2} = \int dt$, since equation is separable.

As if by magic we get

In lecture we used the calculation in the notes	$v(t) = \sqrt{\frac{mg}{k}}\left(\frac{1+Ae^{\alpha t}}{1-Ae^{\alpha t}}\right)$ and $A = \left(\frac{2\alpha T_1 + 1}{2\alpha T_1 - 1}\right)e^{-\alpha T_1}$

First order linear An equation of the form

$$a(x)\frac{dy}{dx} + b(x)y = e(x)$$

is first order linear. $a \neq 0$

Divide by $a(x)$ to get

$$\frac{dy}{dx} + g(x)y = f(x)$$

$g(x) = \frac{b(x)}{a(x)}$, $f(x) = \frac{e(x)}{a(x)}$.

We use an integrating factor.

Notice

$$\frac{d}{dx}\left(e^{\int g(x)\,dx}y\right) = e^{\int g(x)\,dx}y' + g(x) \times e^{\int g(x)\,dx}y$$

If $y' + g(x)y = f(x)$, multiply by $e^{\int g(x)\,dx}$

Then

$$\underbrace{e^{\int g(x)\,dx}y' + g(x)e^{\int g(x)\,dx}y}_{\frac{d}{dx}\left(e^{\int g(x)\,dx}y\right)} = f(x)e^{\int g(x)\,dx}$$

$$\frac{d}{dx}\left(e^{\int g(x)\,dx}y\right) = f(x)e^{\int g(x)\,dx}$$

$$\therefore \quad e^{\int g(x)dx}\, y = \int f(x)\, e^{\int g(x)dx}\, dx$$

We can now find y.

<u>Example</u> $\quad x\dfrac{dy}{dx} - ky = x^2$

or $\quad \dfrac{dy}{dx} - \dfrac{k}{x} y = x \qquad g(x) = -\dfrac{k}{x}$

$$\int g(x)dx = -\int \frac{k}{x}\, dx = -k\ln x$$

$$\therefore \quad e^{\int g(x)dx} = e^{-k\ln x} = e^{\ln x^{-k}} = x^{-k}$$

$\therefore$ Multiply by x^{-k}

$$\underbrace{x^{-k} y' - k x^{-k-1} y}_{\frac{d}{dx}\left(x^{-k} y\right)} = x^{1-k}$$

$$\therefore \quad x^{-k} y = \int x^{1-k}\, dx$$

If $k=2$ $\quad x^{-2} y = \int x^{-1} dx = \ln x + C$

$$\therefore \quad y = x^2(\ln x + C)$$

If $k \neq 2$ $\quad x^{-k} y = \int x^{1-k}\, dx$

$$= \frac{1}{2-k} x^{2-k} + C$$

$$\therefore \quad y = \frac{1}{2-k} x^2 + C x^k$$

<u>Bernoulli Equations</u>. These have the form

$$y' + p(x)y = q(x)y^n \quad n \neq 1$$

Divide by y^n to get

$$y'y^{-n} + p(x)y^{1-n} = q(x)$$

Let $u = y^{1-n}$. Then $\frac{du}{dx} = (1-n)y^{-n}y'$

Thus $\frac{1}{1-n}u' + p(x)u = q(x)$

This is first order linear.

<u>Example</u> $y' + \frac{1}{3x}y = x^2y^4$.

$$u = y^{1-4} = y^{-3}$$

$$\frac{1}{-3}u' + \frac{1}{3x}u = x^2$$

$$\therefore \quad u' - \frac{1}{x}u = -3x^2$$

$$e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

So $\frac{1}{x}u' - \frac{1}{x^2}u = -3x$

or $\left(\frac{1}{x}u\right)' = -3x$

$$\Rightarrow \frac{1}{x}u = -\frac{3x^2}{2} + C$$

Hence $u = -\frac{3x^3}{2} + Cx$

$$y^{-3} = Cx - \frac{3x^3}{2}$$

Solve for y.

Riccati Equations

These have the form

$$m(x)y' + a(x)y + b(x)y^2 = c(x) \quad (*)$$

These can be turned into second order linear equations.

Let $y = A(x)\dfrac{f'}{f}$.

Then $y' = A'(x)\dfrac{f'}{f} + A(x)\dfrac{f''}{f} - A(x)\left(\dfrac{f'}{f}\right)^2$

Divide $(*)$ by $m(x)$ to get

$$y' + \tilde{a}(x)y + \tilde{b}(x)y^2 = \tilde{c}(x)$$

So $$A'\frac{f'}{f} + A\frac{f''}{f} - A(x)\left(\frac{f'}{f}\right)^2 + \tilde{a}(x)A\frac{f'}{f} + \tilde{b}(x)\left(\frac{Af'}{f}\right)^2 = \tilde{c}(x)$$

Let $A(x) = \tilde{b}A^2$, then non-linear terms cancel. This yields the linear DE

$$A'f' + Af'' + \tilde{a}Af' = f(x)\tilde{c}(x)$$

This is second order linear and we will learn how to solve equations of this type.