<u>The Laplace Transform</u> If $f:[0,\infty)\to\mathbb{R}$ is such that

$$|f(t)| \le K e^{at}, \text{ some finite } a, K>0$$

Then the Laplace transform of f,

$$(\mathcal{L}f)(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

whenever the integral converges.

There are other <u>integral transforms</u>.

(1) The Fourier transform

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx$$

(2) The Mellin transform

$$(\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x)\,dx$$

(3) Hankel transform

$$(\mathcal{H}_\nu f)(y) = \int_0^\infty \sqrt{xy}\, J_\nu(xy) f(y)\,dy$$

$$\mathcal{H}_\nu(\mathcal{H}_\nu f) = f$$

These have numerous applications in physics, chemistry, engineering etc.

<u>Example</u> $f(t) = t^\alpha$, $\alpha > -1$

$$(\mathcal{L}f)(s) = \int_0^\infty t^\alpha e^{-st}\,dt$$

Set $st = u$ $du = s\,dt$ or $dt = \frac{1}{s}du$

$t^\alpha = \left(\frac{u}{s}\right)^\alpha = \frac{1}{s^\alpha}u^\alpha$ Hence

$$(\mathcal{L}f) = \int_0^\infty \frac{u^\alpha}{s^\alpha} e^{-u}\frac{du}{s} = \frac{1}{s^{\alpha+1}}\int_0^\infty u^\alpha e^{-u}\,du$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$$

So $\displaystyle\int_0^\infty u^\alpha e^{-u}\,du = \Gamma(\alpha+1)$

$\therefore$ $(\mathcal{L}f)(s) = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.

Recall that $\Gamma(1)=1$, $\Gamma(2) = 1.\Gamma(1) = 1$
$\Gamma(3) = 2\Gamma(2) = 2!$, $\Gamma(4) = 3\Gamma(3) = 3!$
In general $\Gamma(n+1) = n!$ $\quad n = 0, 1, 2, 3 \ldots$

So $\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(t) = \frac{1}{s^2}$, $\mathcal{L}(t^2) = \frac{2!}{s^3}$
etc

Property 1 The Laplace transform is linear. If f, g have Laplace transforms F, G, for $s \geq \sigma$ then

$$\mathcal{L}(af+bg) = aF(s) + bG(s)$$

Proof This is obvious

Theorem Let f be a piecewise continuous function, such that

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt, \qquad s \in \Omega \subset \mathbb{C}$$

Then F is analytic for all $s \in \Omega$

Proof Hard.

Proposition Let f have Laplace transform $F(s)$ then

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s)$$

Proof $\displaystyle\int_0^\infty t f(t)\, e^{-st}\,dt = -\int_0^\infty \frac{d}{ds}(e^{-st})\, f(t)\,dt$

$$= -\frac{d}{ds} F(s)$$

Thus $\mathcal{L}(t f(t)) = -\frac{d}{ds} F(s)$.

Similarly $\mathcal{L}(t^2 f(t)) = \frac{d^2}{ds^2} \int_0^\infty f(t)\, e^{-st}\, dt$

etc.

Example $\mathcal{L}(t^n) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}(1)$

$$= (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = (-1)^n (-1)^n \frac{n!}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}.$$

Example $f(t) = e^{-at}$

$$\int_0^\infty e^{-at} e^{-st}\, dt = \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{-1}{s+a} e^{-(s+a)t}\Big]_0^\infty = \frac{1}{s+a}.$$

Example $f(t) = \sin(at) = \frac{1}{2i}\left(e^{iat} - e^{-iat}\right)$

$$(\mathcal{L}f)(s) = \frac{1}{2i} \mathcal{L}\left(e^{iat} - e^{-iat}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)$$

$$= \frac{1}{2i}\left(\frac{s+ia-(s-ia)}{s^2+a^2}\right) = \frac{1}{2i} \frac{2ia}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2}$$

Example $f(t) = \cos(at) = \frac{1}{2}\left(e^{iat} + e^{-iat}\right)$

$$(\mathcal{L}f) = \frac{1}{2} \mathcal{L}\left(e^{iat} + e^{-iat}\right) = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$$

$$= \frac{1}{2} \frac{s+ia+s-ia}{s^2+a^2} = \frac{s}{s^2+a^2}.$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$
e^{-at}	$\frac{1}{s+a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

This is a table of Laplace transforms