Recall that $(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\,dt$

If $f(t) = \sum_{n=0}^\infty a_n t^n$, then

$$\int_0^\infty \left(\sum_{n=0}^\infty a_n t^n\right) e^{-st}\,dt = \sum_{n=0}^\infty a_n \int_0^\infty t^n e^{-st}\,dt$$

provided the series converges for $t > 0$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}. \quad \text{Thus } \mathcal{L}f = \sum_{n=0}^\infty a_n \frac{n!}{s^{n+1}}.$$

So we have a series expansion for the Laplace transform

Example

$$J_0(t) = \sum_{n=0}^\infty (-1)^n \frac{t^{2n}}{2^{2n} n!\,\Gamma(n+1)}.$$

$(\Gamma(n+1) = n!)$

$$\mathcal{L}J_0 = \sum_{n=0}^\infty (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 s^{2n+1}}$$

Note $\frac{1}{\sqrt{1+s^2}} = \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

So with $x = \frac{1}{s^2}$, $\alpha = -1/2$

$$\frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{s}\left(1 - \frac{1}{2s^2} + \frac{3}{8s^4} - \frac{5}{16}\frac{1}{s^6} + \cdots\right)$$

$$= \sum_{n=0}^\infty (-1)^n \frac{(2n)!}{2^{2n} n!\,\Gamma(n+1)\, s^{2n+1}}$$

$$= \mathcal{L}J_0$$

$$\therefore (\mathcal{L}J_0)(s) = \frac{1}{\sqrt{s^2+1}}$$

Proposition Let f have Laplace transform F, then $\mathcal{L}(e^{-at}f(t)) = F(s+a)$

Proof $\int_0^\infty e^{-at} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(a+s)t} dt$

$= F(s+a)$

Example $\mathcal{L}(e^{-at} J_0(t)) = \dfrac{1}{\sqrt{1+(s+a)^2}}$

Proposition Let $f(t)$ have Laplace transform $F(s)$. Define $f_a(t) = f(at)$, $a>0$.

Then $(\mathcal{L}f_a)(s) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof $\int_0^\infty e^{-st} f(at)\,dt$ $\qquad u = at$, $du = a\,dt$

$= \int_0^\infty e^{-\frac{s}{a}u} f(u) \frac{1}{a} du$

$= \frac{1}{a} F\left(\frac{s}{a}\right)$

Example $\mathcal{L}(J_0(at)) = \dfrac{1}{a} \dfrac{1}{\sqrt{1+(s/a)^2}}$

$= \dfrac{1}{a} \dfrac{a}{\sqrt{a^2+s^2}}$

$= \dfrac{1}{\sqrt{a^2+s^2}}$.

Laplace Transforms of Derivatives

Suppose that $\mathcal{L}f = F$ and f' has a Laplace transform. Assume $e^{-at}f(t) \to 0$ as $t \to \infty$

Then $\int_0^\infty f'(t) e^{-st} dt = f(t) e^{-st}\Big]_0^\infty + s\int_0^\infty f(t) e^{-st} dt$

$= -f(0) + sF(s)$

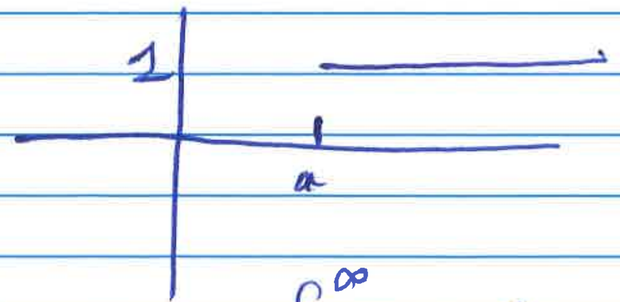
Now
$$\mathcal{L}(f'') = \int_0^\infty f''(t)e^{-st}\,dt = f'(t)e^{-st}\Big]_0^\infty + s\int_0^\infty f'(t)e^{-st}\,dt$$
$$= -f'(0) + s\left(sF(s) - f(0)\right)$$
$$= -f'(0) - sf(0) + s^2F(s)$$

Similarly
$$\mathcal{L}(f''') = -f''(0) - sf'(0) - s^2f(0) + s^3F(s)$$
etc

Aside: If $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx$, then
$$\widehat{f^{(n)}}(y) = (iy)^n\hat{f}(y)$$
$$\mathcal{H}_\nu\left(f'' + \tfrac{1}{r}f\right) = -\nu^2\mathcal{H}_\nu f$$

$KLf = \int_0^\infty f(x)K_{i\xi}(x)\,dx$. Then
$$KL\left(x^2f'' + xf' - x^2f\right) = -\xi^2(KL)f$$

<u>Proposition</u> Let the Heaviside step function be defined by
$$H(x-a) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$$



Then
$$\mathcal{L}\left[H(x-a)f(x-a)\right] = e^{-sa}F(s)$$

<u>Proof</u>
$$\int_0^\infty H(x-a)f(x-a)e^{-sx}\,dx = \int_a^\infty f(x-a)e^{-xs}\,dx$$
$x-a=y$
$$= \int_0^\infty f(y)e^{-(y+a)s}\,ds = e^{-as}F(s)$$
$x=y+a$

The Inverse Laplace Transform

Theorem Suppose that $\mathcal{L}f = 0$. Then $f = 0$ almost everywhere

Corollary If $\mathcal{L}(f) = \mathcal{L}(g)$ all s then $f = g$.

Proof $\mathcal{L}(f) = \mathcal{L}(g) \Rightarrow \mathcal{L}(f-g) = 0 \therefore f-g = 0$

Hence the Laplace transform is unique so its inverse is unique.

The inverse Laplace transform $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}(\mathcal{L}f) = \mathcal{L}(\mathcal{L}^{-1}f) = f$$

Note $\mathcal{L}^{-1}(aF + bG) = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G)$

Example

$$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{1}{n} t^n$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

Example Find $\mathcal{L}^{-1}\left(\frac{s}{(s+2)(s^2+9)}\right)$

$$\frac{s}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}$$

So

$$\frac{s\cancel{(s+2)}}{\cancel{(s+2)}(s^2+9)} = A + \frac{(Bs+C)(s+2)}{s^2+9}.$$

Take $s=-2$ $\quad A = \frac{-2}{4+9} = -\frac{2}{13}$

Then $$\frac{s}{(s+2)(s^2+9)} = -\frac{2}{13}\frac{1}{s+2} + \frac{Bs+C}{s^2+9}$$

Set $s=0$

$$0 = -\frac{2}{13}\cdot\frac{1}{2} + \frac{C}{9}$$

$$\therefore \quad C = \frac{9}{13}.$$

Hence

$$\frac{s}{(s+2)(s^2+9)} = -\frac{2}{13}\frac{1}{s+2} + \frac{Bs}{s^2+9} + \frac{9}{13}\frac{1}{s^2+9}$$

$s=1$ $$\frac{1}{(3)(10)} = -\frac{2}{13}\frac{1}{3} + \frac{B}{10} + \frac{9}{13}\frac{1}{10}$$

$$\Rightarrow \quad B = \frac{2}{13}.$$

Hence $$\frac{s}{(s+2)(s^2+9)} = -\frac{2}{13}\frac{1}{s+2} + \frac{2}{13}\frac{s}{s^2+9} + \frac{9}{13}\frac{3}{3(s^2+9)}$$

$$\therefore \quad \mathcal{L}^{-1}\left(\quad\right) = -\frac{2}{13}e^{-2t} + \frac{2}{13}\cos(3t) + \frac{3\sin(3t)}{13}$$

Example Calculate $\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)(s^2+9)}\right)$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$B = D = 0$, $\quad A = 1/5$, $\quad C = -1/5$

Example $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^4}\right)$

$$\mathcal{L}(H(x-3)f(x-3)) = e^{-3s}F(s)$$

Take $F(s) = \frac{1}{s^4}$ $\quad \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{3!}t^3$

$$\therefore \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^4}\right) = \frac{1}{3!}H(x-3)(x-3)^3$$

$$= \frac{1}{3!}\begin{cases} 0 & x<3 \\ (x-3)^3 & x \geq 3 \end{cases}$$

Inversion by Series If $F(s) = \sum_{n=0}^{\infty} \frac{a_n}{n!\, s^{n+1}}$

Then $\mathcal{L}^{-1}(F) = \sum_{n=0}^{\infty} a_n t^n$.

Example $F(s) = \frac{s}{s^2+a^2}$

$$= \frac{1}{s}\frac{1}{(1+(a/s)^2)}$$

$$= \frac{1}{s}\left(1-\left(\frac{a}{s}\right)^2+\left(\frac{a}{s}\right)^4-\left(\frac{a}{s}\right)^6+-\right)$$

$$= \frac{1}{s}-\frac{a^2}{s^3}+\frac{a^4}{s^5}-\frac{a^6}{s^7}+--$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}-\frac{a^2}{s^3}+\frac{a^4}{s^5}-\frac{a^6}{s^7}+-\right)$$

$$\left(\mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{1}{n!}t^n\right) \quad = 1-\frac{a^2t^2}{2!}+\frac{(at)^4}{4!}-\frac{(at)^6}{6!}+-$$

$$= \cos(at)$$

Example $F(s) = \sin\left(\frac{a}{s}\right)$. We use

$$\sin x = x - \frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\sin(a/s) = \frac{a}{s} - \frac{a^3}{3!\,s^3} + \frac{a^5}{5!\,s^5} - + \cdots$$

$$\mathcal{L}^{-1}\left(\sin\left(\frac{a}{s}\right)\right) = a - \frac{a^3 t^2}{3!\,2!} + \frac{a^5 t^4}{5!\,4!} - \cdots$$

<u>Example</u> $\mathcal{L}^{-1}\left(\tan^{-1}\left(\frac{a}{s}\right)\right)$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \qquad |x| <$$

$$\tan^{-1}\left(\frac{a}{s}\right) = \frac{a}{s} - \frac{a^3}{3s^3} + \frac{a^5}{5s^5} - \frac{a^7}{7s^7} + -$$

$$\mathcal{L}^{-1}\left(\tan^{-1}\left(\frac{a}{s}\right)\right) = a - \frac{a^3 t^2}{2!\,3} + \frac{a^5 t^4}{5.\,4!} - \frac{a^7 t^6}{7.\,6!}$$

$$= a - \frac{a^3 t^2}{3!} + \frac{a^5 t^4}{5!} - \frac{a^7 t^6}{7!} + -$$

$$= \frac{1}{t}\left(at - \frac{(at)^3}{3!} + \frac{(at)^5}{5!} - \frac{(at)^7}{7!} + -\right)$$

$$= \frac{\sin(at)}{t}.$$

So $\mathcal{L}\left(\frac{\sin(at)}{t}\right) = \tan^{-1}\left(\frac{a}{s}\right).$

<u>Example</u> $F(s) = \ln\left(\frac{s+a}{s+b}\right) = \ln(s+a) - \ln(s+b)$

We use

$$\frac{1}{s+a} = \frac{1}{s}\left(\frac{1}{1+s/a}\right) = \frac{1}{s}\left(1 - \frac{a}{s} + \left(\frac{a}{s}\right)^2 - \left(\frac{a}{s}\right)^3 + \cdot\right)$$

$$\int \frac{1}{s+a} = \ln(s+a)$$

So $\ln\left(\frac{s+a}{s+b}\right) = -\frac{(b-a)}{s} + \frac{b^2-a^2}{2s^2} - \frac{(b^3-a^3)}{3s^3} + \cdots$

$$\mathcal{L}^{-1}\left(\quad\right) = -(b-a) + \frac{(b^2-a^2)t}{2} - (b^3-a^3)\frac{t^2}{2!\,3} + \cdots$$

$$= \frac{1}{t}\left(1 - bt + \frac{b^2t^2}{2!} - \frac{b^3t^3}{3!} + \cdots - \left(1 - at + \frac{a^2t^2}{2!} - \frac{a^3t^3}{3!} + \cdots\right)\right)$$

$$= \frac{1}{t}\left(e^{-bt} - e^{-at}\right)$$

$$\therefore\ \mathcal{L}\left(\frac{1}{t}\left(e^{-bt} - e^{-at}\right)\right) = \ln\left(\frac{s+a}{s+b}\right).$$