Convolution This is a particular kind of multiplication for integral transforms

Example

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-iyx} dx$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy$$

This is the Fourier transform convolution.

$$\widehat{(f*g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

For the Laplace Transform. Let $F(s) = \int_0^\infty e^{-st} f(t) dt$

$$G(s) = \int_0^\infty g(s) e^{-st} dt$$

Then
$$F(s)G(s) = \int_0^\infty e^{-sx} g(x) \left(\int_0^\infty f(y) e^{-sy} dy \right) dx$$

$$= \int_0^\infty \int_0^\infty g(x) f(y) e^{-s(x+y)} dx\, dy$$

$x + y = t$, $x = t - u$, $u = y$.

So
$$F(s)G(s) = \int_0^\infty \int_0^t g(t-u) f(u) e^{-st} dt\, du$$

$$= \int_0^\infty e^{-st} \left(\int_0^t g(t-u) f(u) du \right) dt$$

Let
$$\int_0^t g(t-u) f(u) du = \int_0^t f(t-u) g(u) du$$

$$= (f * g)(t)$$

Note It does not matter the order in which we do the integral

So
$$\mathcal{L}(f*g) = (\mathcal{L}f)(\mathcal{L}g).$$

$\int_0^t f(t-u) g(u) du$ is the convolution of f and g

$$(f*g)(t) = (g*f)(t)$$

Example $\mathcal{L}\left(\int_0^t f(u)\,du\right) = \mathcal{L}(f * 1)$

$$= (\mathcal{L} f)(\mathcal{L} 1)$$

$$= \frac{1}{s} F(s)$$

where $\mathcal{L} f = F$

Example $H(s) = \dfrac{1}{s^2(s^2+1)}$

$$= \frac{1}{s^2} \cdot \frac{1}{s^2+1}$$

$\mathcal{L}(t) = \dfrac{1}{s^2}$ $\quad \mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$

So $H(s) = \mathcal{L}\left(\int_0^t (t-u)\sin u\,du\right)$

Thus $\mathcal{L}^{-1} H = \mathcal{L}^{-1}\mathcal{L}\left(\int_0^t (t-u)\sin u\,du\right)$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \int_0^t (t-u)\sin u\,du$$

$$= t - \sin t.$$

Differential Equations and Laplace Transforms The idea is to transform a DE and get something which is easier to solve

Example Solve $y'' + 5y' + 6y = t^2$, $y(0) = 0$, $y'(0) = 2$

Laplace transform both sides.

$$\mathcal{L}(y'' + 5y' + 6y) = \mathcal{L}(y'') + 5\mathcal{L}(y') + 6\mathcal{L}(y) = \mathcal{L}(t^2)$$

Let $Y(s) = \mathcal{L} y$

$$s^2 Y(s) - s y(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) = \frac{2}{s^3}$$

Thus $(s^2+5s+6)Y(s) - (5+s)\,y(0) - y'(0) = \dfrac{2}{s^3}$

$$Y(s) = \frac{2}{s^3(s^2+5s+6)} + \frac{2}{s^2+5s+6}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{2}{s^3(s^2+5s+6)} + \frac{2}{s^2+5s+6}\right)$$

$$\frac{2}{s^2+5s+6} = \frac{A}{s+3} + \frac{B}{s+2}$$

$$\frac{2\cancel{(s+3)}}{(s+2)\cancel{(s+3)}} = A + \frac{B(s+3)}{s+2} \qquad s=-3$$

$$A = \frac{2}{-3+2} = -2$$

$$\frac{2\cancel{(s+2)}}{\cancel{(s+2)}(s+3)} = \frac{A(s+2)}{s+3} + B \qquad s=-2$$

$$\frac{2}{-2+3} = B$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^2+5s+6}\right) = \mathcal{L}^{-1}\left(2\left(\frac{1}{s+2} - \frac{1}{s+3}\right)\right) = 2e^{-2t} - 2e^{-3t}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3(s^2+5s+6)}\right) = 2\int_0^t (t-u)^2\left(e^{-2u} - e^{-3u}\right)du$$

$$y(t) = 2\left(e^{-2t} - e^{-3t}\right) + 2\int_0^t (t-u)^2\left(e^{-2u} - e^{-3u}\right)du$$

$$= \frac{t^2}{6} - \frac{5t}{18} + \frac{19}{108} + \frac{7e^{-2t}}{4} - \frac{52}{27}e^{-3t}.$$