Solve $\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le a$, $t \ge 0$

$u(x,0) = 0$, $\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$, $u(0,t) = u_0$.

Let $\bar{u}(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt$. So

$$\int_0^\infty \frac{\partial u}{\partial t} e^{-st}\,dt = u(x,t)e^{-st}\Big]_0^\infty + s\int_0^\infty u(x,t)e^{-st}\,dt$$

$$= s\bar{u}(x,s)$$

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t)e^{-st}\,dt$$

$$= \frac{d^2\bar{u}}{dx^2}$$

$$\therefore \frac{d^2\bar{u}}{dx^2} = \frac{s}{k}\bar{u}.$$ Constant coefficient

$\lambda^2 - s/k = 0$, $\lambda = \pm\sqrt{s/k}$

$$\bar{u}(x,s) = c_1 e^{-\sqrt{\frac{s}{k}}x} + c_2 e^{\sqrt{\frac{s}{k}}x}.$$

$$= A\cosh\left(\sqrt{\tfrac{s}{k}}x\right) + B\sinh\left(\sqrt{\tfrac{s}{k}}x\right)$$

We need A, B

$\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$. So $\int_0^\infty \frac{\partial u}{\partial x}(a,t)e^{-st}\,dt = 0$

$$\therefore \left.\frac{d\bar{u}}{dx}\right|_{x=a} = 0$$

$$\left.\frac{d\bar{u}}{dx}\right|_{x=a} = A\sqrt{\frac{s}{k}}\sinh\left(\sqrt{\tfrac{s}{k}}x\right)\Big|_{x=a} + B\sqrt{\tfrac{s}{k}}\cosh\left(\sqrt{\tfrac{s}{k}}x\right)\Big|_{x=a}$$

$$= 0$$

$$B = \frac{-A\sinh\left(\sqrt{\frac{s}{k}}a\right)}{\cosh\left(\sqrt{\frac{s}{k}}a\right)}$$

②

$$\bar{u}(0,s) = \int_0^\infty u(0,t)\, e^{-st}\, dt = u_0 \int_0^\infty e^{-st}\, dt = \frac{u_0}{s}.$$

So

$$\bar{u}(0,s) = A\cosh 0 + B\sinh 0 = A = \frac{u_0}{s}.$$

$$B = -\frac{u_0 \sinh\left(a\sqrt{\frac{s}{k}}\right)}{s\cosh\left(a\sqrt{s/k}\right)}.$$

$$\therefore \bar{u}(x,s) = \frac{u_0}{s}\left[\cosh\left(x\sqrt{\frac{s}{k}}\right) - \frac{\sinh\left(a\sqrt{s/k}\right)\sinh\left(x\sqrt{\frac{s}{k}}\right)}{\cosh\left(a\sqrt{s/k}\right)}\right]$$

$$= \frac{u_0}{s}\left[\frac{\cosh\left(x\sqrt{s/k}\right)\cosh\left(a\sqrt{\frac{s}{k}}\right) - \sinh\left(a\sqrt{s/k}\right)\sinh\left(x\sqrt{\frac{s}{k}}\right)}{\cosh\left(a\sqrt{s/k}\right)}\right]$$

$$= \frac{u_0}{s}\,\frac{\cosh\left((x-a)\sqrt{s/k}\right)}{\cosh\left(a\sqrt{s/k}\right)}.$$

$$\therefore u(x,t) = \mathcal{L}^{-1}\left[\frac{u_0}{s}\,\frac{\cosh\left((x-a)\sqrt{\frac{s}{k}}\right)}{\cosh\left(a\sqrt{s/k}\right)}\right]$$

$$= u_0\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\, e^{-\frac{(2n-1)^2\pi^2 k t}{4a^2}} \sin\left(\left(\frac{2n-1}{2a}\right)\pi x\right)\right]$$

<u>The heat equation</u> let $0 \le x \le a$.
$t > 0$

The basic equation we study is

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x), \qquad u(0,t) = u(a,t) = 0$$

Let $u(x,t) = X(x)T(t)$

Then $u(0,t) = X(0)T(t) = 0$

$u(a,t) = X(a)T(t) = 0$

$\therefore X(0) = X(a) = 0$

Then $\frac{1}{k} XT' = X''T$

or $\frac{X''}{X} = \frac{1}{k}\frac{T'}{T}$. Notice that

If $h(x) = g(t)$ all x, t, both are constants

Because $h(0) = g(t)$ So $g(t)$ is constant

$\Rightarrow h(x)$ is constant

Thus $\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = \lambda$, λ a constant

Hence $\frac{d^2X}{dx^2} = \lambda X$, $X(0) = X(a) = 0$.

What is λ? Let $\lambda > 0$. Say $\lambda = m^2$

Then $\frac{d^2X}{dx^2} - m^2 X = 0$

$X(x) = Ae^{mx} + Be^{-mx}$. But

$X(0) = A + B = 0$ $\quad A = -B$

$X(a) = Ae^{ma} + Be^{-ma}$

$= -B(e^{ma} - e^{-ma}) = 0$

$\therefore B = 0 \Rightarrow A = 0 \Rightarrow X = 0$

So λ is not positive

If $\lambda = 0$ $\quad X'' = 0 \quad \therefore X = Ax + B$

But $X(0) = B = 0$, $X(a) = Aa = 0$

$\therefore A = 0$

$\therefore X = 0$ $\quad$ So $\lambda \neq 0$

$\lambda < 0$, $\lambda = -m^2$

$X(x) = A\cos(mx) + B\sin(mx)$

$X(0) = A = 0$. $X(a) = B\sin(ma) = 0$

$\therefore$ $ma = n\pi$, $n = 1, 2, 3.$

$$\lambda = -m^2 = -\frac{n^2\pi^2}{a^2}$$

and we have solutions

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

Then $\frac{dT}{dt} = -\lambda k T$

$$\Rightarrow T(t) = Ce^{-\frac{n^2\pi^2 kt}{a^2}}$$

$u(x,t) = C\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{n^2\pi^2 kt}{a^2}}$. This is a solution of the heat equation

But we need $u(x,0) = f(x)$

Idea:

Write $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n^2\pi^2 kt}{a^2}}$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$