The problem is as follows. We have a function f on $[0,1]$. We wish to write

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

for $x \in [0,1]$. How do we find the numbers C_n? We use the following integral

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \end{cases}$$

If $n = m$ $\quad \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \int_0^1 \sin^2(n\pi x)\,dx$

Now $\sin^2(n\pi x) + \cos^2(n\pi x) = 1 \quad \therefore \sin^2(n\pi x) = 1 - \cos^2(n\pi x)$

$$\cos(2n\pi x) = \cos(n\pi x + n\pi x)$$

$$= \cos(n\pi x)\cos(n\pi x) - \sin(n\pi x)\sin(n\pi x)$$

$$= \cos^2(n\pi x) - \sin^2(n\pi x)$$

S $\quad = \cos^2(n\pi x) - (1 - \cos^2(n\pi x))$

$$= 2\cos^2(n\pi x) - 1$$

$$\cos^2(n\pi x) = \tfrac{1}{2}(1 + \cos(2n\pi x))$$

$$\sin^2(n\pi x) = \tfrac{1}{2}(1 - \cos(2n\pi x))$$

$$\therefore \int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}\int_0^1 (1 - \cos(2n\pi x))\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2n\pi}\sin(2n\pi x)\right]_0^1$$

$$= \frac{1}{2}$$

Similarly $\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = 0$ $\quad n \neq m$

[Expand $\sin((n+m)\pi x)$]

If $$f(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

Then
$$f(x)\sin(\pi x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\sin(\pi x)$$

Thus

$$f(x)\sin(\pi x) = C_1 \sin^2(\pi x) + C_2 \sin(2\pi x)\sin(\pi x) + C_3 \sin(3\pi x)\sin(\pi x) + \cdots$$

Hence
$$\int_0^1 f(x)\sin(\pi x)\,dx = C_1 \int_0^1 \sin^2(\pi x)\,dx + C_2 \int_0^1 \sin(2\pi x)\sin(\pi x)\,dx + C_3 \int_0^1 \sin(3\pi x)\sin(\pi x)\,dx + \cdots$$
$$= \frac{1}{2}C_1$$
$$\therefore\ C_1 = 2\int_0^1 f(x)\sin(\pi x)\,dx$$

Similarly
$$\int_0^1 f(x)\sin(2\pi x)\,dx = C_1 \int_0^1 \sin(\pi x)\sin(2\pi x)\,dx + C_2 \int_0^1 \sin^2(2\pi x)\,dx + C_3 \int_0^1 \sin(3\pi x)\sin(2\pi x)\,dx + \cdots$$
$$= \frac{1}{2}C_2$$
$$\therefore\ C_2 = 2\int_0^1 f(x)\sin(2\pi x)\,dx$$

In general
$$C_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

The numbers C_n are called the Fourier sine coefficients. The series

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

is a Fourier sine series

Given this the solution of

$$\frac{1}{k} u_t = u_{xx} \qquad x \in [0,1]$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = f(x)$$

is

$$u(x,t) = 2 \sum_{n=1}^{\infty} \left\{ \int_0^1 f(y) \sin(n\pi y)\, dy \right\} \sin(n\pi x) \times e^{-n^2 k \pi t}$$

Fourier Series We will work on the interval $(-\pi, \pi)$. The following hold

$$\int_{-\pi}^{\pi} \cos(nx)\, dx = \int_{-\pi}^{\pi} \sin(nx)\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx = 0, \quad \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin^2(nx)\, dx = \int_{-\pi}^{\pi} \cos^2(nx)\, dx = \pi.$$

We would like

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}$$

This is a Fourier series (probably)

How do we find a_n, b_n? As before

$$f(x)\sin(mx) = a_0 \sin(mx) + \sum_{n=1}^{\infty} a_n \cos(nx)\sin(mx) + \sum_{n=1}^{\infty} b_n \sin(nx)\sin(mx)$$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \cancel{\int_{-\pi}^{\pi} a_0 \sin(mx)\,dx}^{\,0} + \cancel{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \sin(mx)\cos(nx)\,dx}^{\,0} + \cancel{\int_{-\pi}^{\pi} b_1 \sin x \sin(mx)\,dx} + \dots \int_{-\pi}^{\pi} b_m \sin^2(mx)\,dx + \dots$$

$$= \pi b_m$$

$$\therefore \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

Similarly

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx, \quad m \geq 1.$$

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos(nx)\,dx + \sum \int_{-\pi}^{\pi} b_n \sin(nx)\,dx$$

$$= 2\pi a_0$$

or $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

These are called the Fourier coefficients.

Does this work and for what functions

We will consider periodic functions, with period 2π. We consider functions on $(-\pi, \pi)$ and extend to $\mathbb{R}$ by setting

$$f(x+2\pi) = f(x) \quad \text{all } x$$

Example $\quad f(x) = x \quad -\pi < x < \pi$

$$f(x+2\pi) = f(x)$$