Example $f(x) = x, \ -\pi < x < \pi$

$f(x+2\pi) = f(x)$ all x

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{1}{2}x^2\right]_{-\pi}^{\pi} = 0$$

Similarly $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left[\frac{-x\cos(nx)}{n}\right]_{-\pi}^{\pi} + \cancel{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(nx)}{n}\,dx}^{\,0}$$

$$= \frac{-1}{\cancel{\pi}}\left(\frac{\cancel{\pi}\cos(n\pi)}{n} - \frac{(-\pi)\cos(-n\pi)}{n}\right)$$

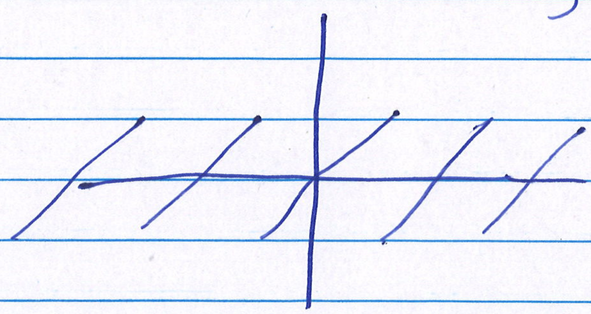
$$= \frac{-2\cos(n\pi)}{n} = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

The Fourier series for f is

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

Notice there are no cosine terms

See notes for graph of $2\sum_{n=1}^{100}\frac{(-1)^{n+1}}{n}\sin(nx)$



<u>Example</u> $f(x) = x^2, \quad -\pi < x < \pi$

$f(x+2\pi) = f(x)$ all x

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right) = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\cancel{\left[\frac{x^2\sin(nx)}{n}\right]_{-\pi}^{\pi}}^{\,0} - \int_{-\pi}^{\pi} 2x\,\frac{\sin(nx)}{n}\,dx\right)$$

$$= -\frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= \frac{-2}{n}\,\frac{2(-1)^{n+1}}{n} = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0$$

F.S. $\displaystyle \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$

p 128 for graph of $\displaystyle\sum_{n=0}^{100} a_n\cos(nx)$

<u>Example</u> $f(x) = x^3 + 3x^2 - 25\sin^2 x \quad x \in (-\pi, \pi)$

$f(x+2\pi) = f(x), \quad x \in \mathbb{R}$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^3 + 3x^2 - 25\sin^2 x)\,dx$$

$$= \frac{1}{2}(2\pi^2 - 25)$$

$n=2 \quad \displaystyle\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 x\cos(2x)\,dx = -\frac{1}{2}$

$n \neq 2 \quad \displaystyle\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 x\cos(nx)\,dx = 0$ etc

etc. F S is

$$\frac{1}{2}(2\pi^2 - .25) + \frac{25}{2}\cos(2x)$$

$$+ \sum_{n=1}^{\infty}\left(12\frac{(-1)^n}{n^2}\cos(nx) + \frac{2(-1)^{n+1}(n^2\pi^2-6)}{n^3}\sin(nx)\right)$$

p128, p129

<u>ODD and Even functions</u> If $f(x)$ is odd

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n \geq 1$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \int_0^{\pi} f(x)\cos(nx)\,dx\right)$$

$x = -u$

$$= \frac{1}{\pi}\left(\int_{\pi}^{0} f(-u)\cos(-nu)(-du)\right.$$

$$\left. + \int_0^{\pi} f(x)\cos(nx)\,dx\right)$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} f(-u)\cos(nu)\,du + \int_0^{\pi} f(x)\cos(nx)\,dx\right)$$

$u \to x$

$$= \frac{1}{\pi}\left(\int_0^{\pi}\left(-f(x)\right)\cos(nx)\,dx + \int_0^{\pi} f(x)\cos(nx)\,dx\right)$$

$$= 0$$

Similarly $a_0 = 0$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \int_0^{\pi} f(x)\sin(nx)\,dx\right)$$

$x = -u$

$$= \frac{1}{\pi}\left(\int_{\pi}^{0} f(-u)\sin(-nu)(-du)\right.$$

$$\left. + \int_0^{\pi} f(x)\sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

If f is even, a similar calculation gives $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx$$

Convergence of Fourier Series.

__Lemma__ Riemann–Lebesgue Lemma

If f is differentiable on $(-\pi,\pi)$ $a_n, b_n \to 0$ as $n \to \infty$.

Weierstrass M test. If $|f_n(x)| \le M_n$ all $x \in X \subseteq \mathbb{R}$, and $\sum_{n=1}^\infty M_n < \infty$. Then $\sum_{n=1}^\infty f_n(x)$ converges uniformly on X.

__Example__ $f(x) = x^2$ has F. S

$$\frac{\pi^2}{3} + 4\sum_{n=1}^\infty (-1)^n \frac{1}{n^2}\cos(nx)$$

$$|f_n(x)| = \left|\frac{4}{n^2}(-1)^n\cos(nx)\right| \le \frac{4}{n^2}.$$

and $\sum_{n=1}^\infty \frac{4}{n^2} = 4\frac{\pi^2}{6} = \frac{2\pi^2}{3} < \infty$

So $\sum_{n=1}^\infty f_n(x)$ converges uniformly to a continuous function.

In fact if $f(x)$ is twice continuously differentiable, the coefficients decay at least as fast as $\frac{1}{n^2}$ (assuming f is 2π periodic)

$$\pi b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \left(\cancel{-f(x)\frac{\cos(nx)}{n}\Big|_{-\pi}^{\pi}}^{\,0} \quad \text{as } f(\pi)=f(-\pi) \right.$$

$$\left. + \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\frac{\cos(nx)}{n}\,dx \right)$$

$$= \cancel{\frac{1}{n^2}f'(x)\frac{\sin(nx)}{n^2}\Big|_{-\pi}^{\pi}}^{\,0} - \frac{1}{n^2}\int_{-\pi}^{\pi} f''(x)\sin(nx)\,dx$$

$$= -\frac{1}{n^2}\int_{-\pi}^{\pi} f''(x)\sin(nx)\,dx$$

$$|\pi b_n| \le \frac{1}{n^2}\int_{-\pi}^{\pi} |f''(x)|\,|\sin(nx)|\,dx$$

$$\le \frac{1}{n^2}\int_{-\pi}^{\pi} |f''(x)|\,dx = \frac{A}{n^2}$$

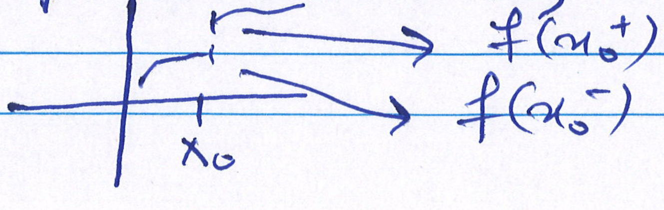
$$A = \int_{-\pi}^{\pi} |f''(x)|\,dx < \infty$$

Similarly for a_n. This suggests that the Fourier for a "nice" function should converge. But does it give the function back?

<u>Dirichlet's Theorem</u> Let f be a piecewise differentiable function on $(-\pi, \pi)$ then the Fourier series of f converges to f at any x_0 where f is continuous and to

$$\frac{1}{2}\left(f(x_0^+) + f(x_0^-)\right)$$ where x_0 is a jump discontinuity.

That is, the F.S converges to the average of left and right values of f at x_0.



Example $f(x) = x \quad , \quad x \in (-\pi, \pi)$

$f(x+2\pi) = f(x)$ all x

By Dirichlet

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \qquad x \in (-\pi, \pi)$$

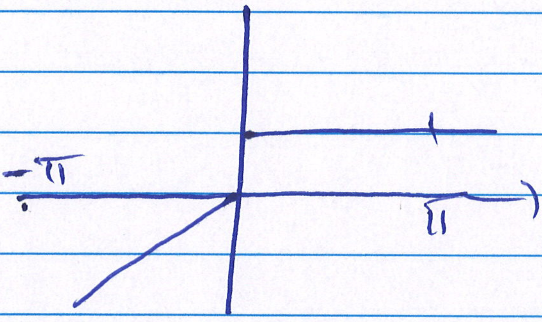
Take $x = \frac{\pi}{2}$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

This is Leibnitz's series. It converges very slowly.

Example $f(x) = \begin{cases} 1 & 0 \le x < \pi \\ x & -\pi < x < 0 \end{cases}$



$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} dx\right)$$

$$= \frac{2-\pi}{4}$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\cos(nx)\,dx + \int_{0}^{\pi} \cos(nx)\,dx\right)$$

$$= -\frac{(-1 + (-1)^n)}{n^2\pi}$$

$$b_n = -\frac{(-1 + (-1)^n + (-1)^n\pi)}{n\pi}$$

The F.S. converges to f at $x \neq 0$ and $1/2$ at $x = 0$

<u>Parseval's Theorem</u>. Let f be piecewise differentiable on $(-\pi, \pi)$. Let a_n, b_n be the Fourier coefficients.

Then $$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

<u>Example</u> $f(x) = x \qquad x \in (-\pi, \pi)$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = 0 + \sum_{n=1}^{\infty}\left(\left(\frac{2(-1)^n}{n}\right)^2 + 0\right)$$

$$= 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

So $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2 dx$$

$$= \frac{1}{4\pi}\left[\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{4\pi}\,\frac{2\pi^3}{3}$$

$$= \frac{\pi^2}{6}.$$

<u>Why?</u> Let $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$

Then $(f(x))^2 = a_0^2 + a_1^2\cos^2 x + a_2^2\cos^2(2x) + \ldots$
$+ b_1^2\sin^2 x + b_2^2\sin^2(2x)$
$+$ cross terms

cross terms are $\cos(nx)\sin(mx)$
$\sin(nx)\sin(mx)$, $\cos(nx)\cos(mx)$ etc

$$\int_{-\pi}^{\pi} \text{Cross terms} = 0$$

$$\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \int_{-\pi}^{\pi}\cos^2(nx)\,dx = \pi$$

$$\int_{-\pi}^{\pi} a_0^2 = 2\pi a_0^2$$

So $$\int_{-\pi}^{\pi} (f(x))^2 dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$
which gives the result