Example Calculate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

We use Parseval's Theorem, $f(x) = x^2$ $-\pi < x < \pi$.

$$b_n = 0, \quad a_n = \frac{4(-1)^n}{n^2}, \quad a_0 = \frac{\pi^2}{3}.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2$$

So $$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = 2\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty} \frac{16}{n^4}.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{1}{\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Hence $$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16}\left(\frac{2\pi^4}{5} - \frac{2\pi^4}{9}\right)$$
$$= \frac{\pi^4}{8}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{8}\left(\frac{9-5}{45}\right)$$
$$= \frac{\pi^4}{90}.$$

If we take $f(x) = x^3$ we can show (p 137). that $\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$.

The values of $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ $k = 1, 2, 3, \ldots$ are all known. (Found by Euler)

However $\sum_{n=1}^{\infty}\frac{1}{n^3}$, $\sum_{n=1}^{\infty}\frac{1}{n^5}$, $\sum_{n=1}^{\infty}\frac{1}{n^7}$ etc are all unknown

In 1978, Apery showed that $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is irrational. The series is known as Apery's constant

Fourier Series on $(-L, L)$. We want to expand f on $(-L, L)$ with period $2L$. Let $T = 2L$. Define

$$g(x) = f\left(\frac{T}{2\pi}x\right). \quad g(x+2\pi) = f\left(\frac{T(x+2\pi)}{2\pi}\right)$$
$$= f\left(\frac{Tx}{2\pi} + T\right) = f\left(\frac{Tx}{2\pi}\right)$$
$$= g(x)$$

We can expand g as a F.S on $(-\pi, \pi)$

$$(*) \quad f\left(\frac{Tx}{2\pi}\right) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{Tx}{2\pi}\right)dx. \text{ Let } t = \frac{Tx}{2\pi},\ dt = \frac{T}{2\pi}dx$$

$$a_0 = \frac{\pi}{2\pi L}\int_{-L}^{L} f(t)\,dt = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Tx}{2\pi}\right)\cos(nx)\,dx$$
$$= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

Similarly $\quad b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$

$(*)$ Becomes

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right)$$

on $(-L, L)$

Parseval's Theorem becomes

$$\frac{1}{L}\int_{-L}^{L}|f(t)|^2\,dt = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Example $f(t)=t^2$, $-1<t<1$, $f(t+2)=f(t)$

$b_n=0$

$$a_0=\frac{1}{2}\int_{-1}^{1}t^2\,dt=\frac{1}{3}$$

$$a_n=\int_{-1}^{1}t^2\cos(n\pi t)\,dt=\frac{4(-1)^n}{n^2\pi^2}$$

$$t^2=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2\pi^2}\cos(n\pi t)$$

Half Range Expansion We want to expand a function on $[0,L)$ The idea is to extend f backwards to a function on $(-L,L)$. We have the odd extension

$$f_o(x)=\begin{cases} f(x) & x\in[0,L) \\ -f(-x) & x\in(-L,0). \end{cases}$$

The even extension is

$$f_E(x)=\begin{cases} f(x) & x\in[0,L) \\ f(-x) & x\in(-L,0) \end{cases}$$

Expand f_o on $(-L,L)$. Since f_o is odd, the cosine terms are zero

$$a_n=\frac{1}{L}\int_{-L}^{L}f_o(x)\cos\left(\frac{n\pi x}{L}\right)dx=0.$$

and $$b_n=\frac{1}{L}\int_{-L}^{L}f_o(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{1}{L}\left(\underbrace{\int_{-L}^{0}-f(-x)\sin\left(\frac{n\pi x}{L}\right)dx}_{x\to -x}+\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$\int_{-L}^{0} -f(-x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\therefore \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

We can write $f_o(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

for $x \in (-L, L)$. But for $x \in (0, L)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

For f_E

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$b_n = 0$$

We get

$$f(x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) + a_0$$

for $x \in [0, L)$

Example $\quad f(x) = x \qquad x \in [0,1)$

The sine series is given by

$$b_n = 2\int_{0}^{1} x\sin(n\pi x)\,dx$$

$$= 2\,\frac{(-1)^{n+1}}{n\pi}$$

Plot of $\sum_{n=1}^{100} b_n$ on p 141