Complex Fourier series.

If we have a function on $(-\pi, \pi)$ we can also expand as an exponential Fourier series.

We define

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi$$

These are the Fourier coefficients

Then if f is differentiable

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

Similar results hold as for the regular series,

For Example if $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$

Then $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|^2$

$\displaystyle\lim_{n\to\infty} |\hat{f}(n)| = 0$, if f is continuous.

(p143)

We will use this later

Return to PDEs We solved

$$u_t = k u_{xx}, \quad 0 \leq x \leq 1, \quad t > 0$$

$u(0,t) = u(1,t) = 0$, $u(x,0) = f(x)$, we now know that if f' exists

$$u(x,t) = 2\sum_{n=1}^{\infty} \left\{ \int_0^1 f(y) \sin(n\pi y)\, dy \right\} \sin(n\pi x) e^{-n^2\pi^2 k t}$$

If $k=1$, $f(x) = x(1-x)$,

$$b_n = 2\int_0^1 y(1-y)\sin(n\pi y)\,dy = \frac{4(1-(-1)^n)}{n^3\pi^3}$$

So

$$u(x,t) = \sum_{n=1}^{\infty} 4\frac{(1-(-1)^n)}{n^3\pi^3}\sin(n\pi x)\,e^{-n^2\pi^2 t}$$

$$= \sum_{n=0}^{\infty} \frac{8}{(2n+1)^3\pi^3}\sin((2n+1)\pi x)\,e^{-(2n+1)^2\pi^2 t}$$

We only need a few terms to get a very accurate solution

<u>Example</u> $u_t = u_{xx}$, $x\in[0,L]$, $t\geq 0$

$$u(x,0) = f(x) \quad u_x(0,t) = u_x(L,t) = 0.$$

We use separation of variables.

$$u(x,t) = X(x)T(t), \quad X'(0) = X'(L) = 0$$

We have

$$X''T = XT' \quad \text{or}$$

$$\frac{X''}{X} = \frac{T'}{T} = \lambda, \text{ a constant.}$$

Let $\lambda = \omega^2 > 0$. Then $X'' - \omega^2 X = 0$

or $X(x) = Ae^{\omega x} + Be^{-\omega x}$

$$X'(x) = \omega(Ae^{\omega x} - Be^{-\omega x})$$

So

$$X'(0) = \omega(A-B) = 0 \quad \therefore A = B$$

$$X'(L) = \omega A(e^{\omega L} - e^{-\omega L}) = 0$$

$$\therefore A = 0 \Rightarrow B = 0.$$

and this gives $X = 0$

So λ cannot be positive.

$$X'' + \omega^2 X = 0$$ ③

If $\lambda = -\omega^2$, $X(x) = A\cos(\omega x) + B\sin(\omega x)$

$X'(0) \stackrel{!}{=} -A\omega\sin 0 + B\omega\cos 0 = 0$

$\therefore B = 0$

$X'(L) \stackrel{!}{=} -\omega A\sin(\omega L) = 0$

$\therefore \omega L = n\pi, \quad n = 0, 1, 2, 3 \ldots$

$$\omega = \frac{n\pi}{L}.$$

So $$X_n(x) = A\cos\left(\frac{n\pi x}{L}\right)$$

$$\lambda = -\frac{n^2\pi^2}{L^2}.$$

Solution satisfying the boundary conditions are satisfied by

$$u_n(x,t) = B_n\cos\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2}{L^2}t}.$$

$(T' = \lambda T)$ To satisfy $u(x,0) = f(x)$

we write

$$u(x,t) = \sum_{n=0}^{\infty} B_n\cos\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2 t}{L^2}}.$$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} B_n\cos\left(\frac{n\pi x}{L}\right)$$

$$B_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$B_n = \frac{2}{L}\int_0^L f(y)\cos\left(\frac{n\pi y}{L}\right)dy$$

We can have mixed B.C. eg.

$$\alpha\, u(0,t) + \beta\, u_x(0,t) = 0$$

$$\alpha_2\, u(L,t) + \beta_2\, u_x(L,t) = 0$$

The Wave Equation Solve

$$\frac{1}{c^2} u_{tt} = u_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

We use separation of variables

$u(x,t) = X(x)T(t)$. Then $X(0) = X(L) = 0$

we have

$$X''T = \frac{1}{c^2} X T''$$

or $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T} = \lambda$, a constant

Hence $X'' = \lambda X$. Set $\lambda = k^2 > 0$

$$X'' - k^2 X = 0 \qquad X(0) = X(L) = 0$$

Same problem as before. $X = 0$ is only solution. $\lambda = 0$ gives $X = 0$.

If

$\lambda = -k^2$. $\quad X_n(x) = A \sin\left(\dfrac{n\pi x}{L}\right)$. $\quad \lambda = \dfrac{-n^2\pi^2}{L^2}$

Then $\dfrac{d^2T}{dt^2} = -\dfrac{n^2\pi^2 c^2}{L^2} T$

For each n

$$T_n(t) = D_n \cos\left(\frac{n\pi c}{L} t\right) + B_n \sin\left(\frac{n\pi c}{L} t\right)$$

$$u_n(x,t) = \sin\left(\frac{n\pi}{L} x\right) T_n(t)$$

Then we have a solution by super position

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L} t\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c}{L} t\right)$$

So $u(x,0)=f=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$ as $\sin 0=0$, $\cos 0=1$

$$\therefore C_n = \frac{2}{L}\int_0^L f(y)\sin\left(\frac{n\pi}{L}y\right)$$

Next

$$u_t(x,0) = \sum_{n=1}^{\infty} -C_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi c}{L}\sin(0) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi c}{L}\cos 0$$

$$= g(x)$$

$$\therefore g(x) = \sum_{n=1}^{\infty} A_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)$$

So $A_n \dfrac{n\pi c}{L} = \dfrac{2}{L}\displaystyle\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)dy$

or $A_n = \dfrac{2}{n\pi c}\displaystyle\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)dy$

So

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(y)\sin\left(\frac{n\pi}{L}y\right)dy\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L}t\right)$$
$$+\sum_{n=1}^{\infty}\left(\frac{2}{n\pi c}\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)dy\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c}{L}t\right)$$

Example Take $L=1$, $f(x)=x(1-x)$, $g(x)=x$

$$A_n = 2\int_0^1 y(1-y)\sin(n\pi y)\,dy = \frac{4(1-(-1)^n)}{n^3\pi^3}$$

$$B_n = \frac{2}{n\pi c}\int_0^1 y\sin(n\pi y)\,dy = \frac{2(-1)^{n+1}}{n^2\pi^2 c}.$$

Take $c=1$. See p148

<u>Laplace's Equation</u> This is

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0$$

Δ is the Laplacian.

<u>Theorem</u> (The maximum principle). Let $\Delta u = 0$ on $\Omega \subset \mathbb{R}^n$. Then the maximum and minimum values of the solution occur on the boundary of Ω, written $\partial\Omega$.

<u>Theorem</u> Consider the Dirichlet problem $\Delta u = 0$, $x \in \Omega$, $\Omega \subset \mathbb{R}^n$,

$$u\big|_{\partial\Omega} = f,$$

If there is a solution, it is unique.

<u>Proof</u> Let u, v both solve the problem. Thus $\Delta u = \Delta v = 0$

$$u\big|_{\partial\Omega} = f, \quad v\big|_{\partial\Omega} = f$$

$$\Delta(u-v) = 0, \text{ and } u\big|_{\partial\Omega} - v\big|_{\partial\Omega} = f - f = 0$$

So $\max(u-v) = \min(u-v) = 0$

Thus $u - v = 0$ or $u = v$.

<u>Example</u> Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$

$$u(0,y) = 0, \; u(a,y) = 0, \; u(x,b) = 0$$

$$u(x,0) = f(x)$$