Solve $u_{xx} + u_{yy} = 0 \quad 0 \le x \le a, \quad 0 \le y \le b$

$u(0,y) = 0, \; u(a,y) = 0, \; u(x,b) = 0,$

$u(x,0) = f(x)$

We use separation of variables.

$u(x,y) = X(x)Y(y)$.

So $u_{xx} + u_{yy} = X''Y + XY'' = 0$

Hence

$$X''Y = -XY'' \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \text{ a constant}$$

We have $X(0) = X(a) = 0 \qquad Y(b) = 0.$

Thus we have 3 possibilities

(1) $\lambda = k^2 > 0$

Then $X'' = k^2 X$. Solution is

$X(x) = Ae^{kx} + Be^{-kx}$

So $X(0) = A + B = 0 \quad \therefore A = -B$

and $X(a) = A(e^{ka} - e^{-ka}) = 0$

$\therefore A = 0$

Thus k is not positive.

(2) $\lambda = 0$. Gives $X(x) = Ax + B$

$X(0) = B = 0. \qquad X(a) = Aa = 0 \therefore A = 0$

$\therefore X = 0$

Hence $\lambda \neq 0$

(3) $\lambda = -k^2 \qquad X = A\cos(kx) + B\sin(kx)$

Now $X(0) = A = 0, \quad X(a) = B\sin(ka) = 0$

$\therefore ka = n\pi, \quad n = 1, 2, 3, 4, \ldots$

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right)$$

Also $\lambda = -\dfrac{n^2\pi^2}{a^2}$.

and $Y'' = \dfrac{n^2\pi^2}{a^2} Y, \quad Y(b) = 0$

$$\therefore \; Y(y) = C\cosh\left(\frac{n\pi y}{a}\right) + D\sinh\left(\frac{n\pi y}{a}\right)$$

Next $Y(b) = C\cosh\left(\frac{n\pi b}{a}\right) + D\sinh\left(\frac{n\pi b}{a}\right) = 0$

$$\therefore \; C = -\frac{D\sinh\left(\frac{n\pi b}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \quad \left(= -D\tanh\left(\frac{n\pi b}{a}\right)\right)$$

or $Y(y) = D\sinh\left(\frac{n\pi y}{a}\right) - \frac{D\sinh\left(\frac{n\pi b}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)}\cosh\left(\frac{n\pi y}{a}\right)$

$$= D\left[\frac{\cosh\left(\frac{n\pi b}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) - \sinh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)}\right]$$

$$= \frac{-D}{\cosh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi(b-y)}{a}\right) \quad \left[\text{Double angle formula}\right]$$

$$\therefore \; u(x,y) = \frac{\overline{D}B_n}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)$$

$\overline{D} = -D.$

To obtain $u(x,0) = f(x)$ we take a sum

$$u(x,y) = \sum_{n=1}^{\infty} A_n \frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \tanh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$\therefore \; A_n\tanh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(z)\sin\left(\frac{n\pi z}{a}\right)dz$$

$$\therefore \; u(x,y) = \sum_{n=1}^{\infty}\left[\frac{2}{a}\frac{1}{\tanh\left(\frac{n\pi b}{a}\right)}\int_0^a f(z)\sin\left(\frac{n\pi z}{a}\right)\right]\frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \times \sin\left(\frac{n\pi x}{a}\right)$$

$a = b = 1$, $f(x) = x(1-x)$. See p151

Example

Solve the Laplace equation on a disc $D_R = \{(x,y) \in \mathbb{R}^2, \; x^2+y^2 < R^2\}$

u on boundary is $f(r)$

Assume u is finite as $r \to 0^+$

Put $x = r\cos\theta$, $y = r\sin\theta$. Chain rule gives

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Let

$$u(r,\theta) = V(r)\,\Phi(\theta)$$

Then

$$\Phi\left(V'' + \frac{1}{r}V'\right) + \frac{1}{r^2}\Phi'' V = 0$$

or $\frac{1}{V}\left(V'' + \frac{1}{r}V'\right) = -\frac{1}{r^2}\frac{\Phi''}{\Phi}$

So $\frac{r^2}{V}\left(V'' + \frac{1}{r}V'\right) = -\frac{\Phi''}{\Phi} = \lambda = n^2$

Thus $r^2 V'' + rV' - n^2 V = 0$, $\Phi'' = -n^2 \Phi$

Let $V = r^a$, then $r^2 a(a-1)r^{a-2} + rar^{a-1} - n^2 r^a = 0$

$\therefore a^2 - n^2 = 0$ or $a = \pm n$

So

$V(r) = Ar^n + Br^{-n}$. Since $\lim_{r \to 0^+} u(r,\theta)$ is finite, $B = 0$

$$\therefore V(r) = Ar^n$$

$\Phi'' = -n^2\Phi$, $\therefore \Phi = Ce^{in\theta} + De^{-in\theta}$

or

$$u(r,\theta) = Ar^n\left(Ce^{in\theta} + De^{-in\theta}\right), \quad n = 0, 1, \ldots$$

Since $u(R,\theta) = f(\theta)$, let

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n \left(A_n e^{in\theta} + B_n e^{-in\theta}\right)$$

Let $B_n = A_{-n}$ $n \geq 1$, $B_0 = 0$

$$\therefore u(r,\theta) = \sum_{n=0}^{\infty} r^n \left(A_n e^{in\theta} + A_{-n} e^{-in\theta}\right)$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} A_n e^{in\theta} \qquad 0 \leq \theta \leq 2\pi.$$

So

$$u(R,\theta) = \sum_{n=-\infty}^{\infty} R^{|n|} A_n e^{in\theta} = f(\theta)$$

$$A_n = \frac{1}{2\pi R^{|n|}} \int_0^{\infty} f(\varphi) e^{-in\varphi} d\varphi$$

$$\therefore u(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|n|} \left(\int_0^{2\pi} f(\varphi) e^{-in\varphi} d\varphi\right) e^{in\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) \underbrace{\left[\sum_{n=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|n|} e^{in(\theta-\varphi)}\right]}_{P} d\varphi$$

Now P is a geometric series

$$P = 1 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n e^{in(\theta-\varphi)} + \sum_{n=-\infty}^{-1} \left(\frac{r}{R}\right)^n e^{in(\theta-\varphi)}$$

$$= \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\varphi) + r^2}$$ after some algebra. ($e^{i\theta} = \cos\theta + i\sin\theta$)

$$\therefore u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)}{R^2 - 2rR\cos(\theta-\varphi) + r^2} f(\varphi) d\varphi$$

This is Poisson's formula.