Exact Equations These have the form

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad (*)$$

$$\left[\text{or } \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}\right]$$

The equation is exact if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

In this case there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial y} = Q, \quad \frac{\partial F}{\partial x} = P.$$

[If such an F exists

This means $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial P}{\partial y}$]

Then the equation $F(x,y) = c$ defines an implicit solution of $(*)$.

Example Solve $(\underbrace{3x^2y - 2y^3 + 3}_{P})dx + (\underbrace{x^3 - 6xy^2 + 2y}_{Q})dy = 0$

$$\frac{\partial P}{\partial y} = 3x^2 - 6y^2, \quad \frac{\partial Q}{\partial x} = 3x^2 - 6y^2$$

So $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Equation is exact. There is an F such that

$$\frac{\partial F}{\partial x} = P = 3x^2y - 2y^3 + 3$$

$$\therefore \quad F = x^3y - 2xy^3 + 3x + h(y) \quad (A)$$

$$\frac{\partial F}{\partial y} = Q = x^3 - 6xy^2 + 2y$$

From (A) $\frac{\partial F}{\partial y} = x^3 - 6xy^2 + h'(y)$

$$= x^3 - 6xy^2 + 2y$$

Thus $h'(y) = 2y$. or $h(y) = y^2 + c$

Hence $F(x,y) = x^3 - 6xy^2 + 6y = C$
is an implicit solutions.
In theory, if P, Q are "reasonable"
then there is a function $I(x,y)$
such that
$$I(x,y)P(x,y)\,dx + I(x,y)Q(x,y)\,dy = 0$$
is exact

<u>Homogeneous Equations</u> A function is
homogeneous of degree k if
$$f(tx_1, .., tx_n) = t^k f(x_1, .., x_n)$$

<u>Example</u> $f(x,y,z) = x^2y^3z^4$

Then $f(tx, ty, tz) = t^9 x^2y^3z^4$.

<u>Theorem</u> If $\dfrac{dy}{dx} = \dfrac{P(x,y)}{Q(x,y)}$, P, Q are
homogeneous of the same degree, then
setting $y = xv(x)$ produces a separable
equation

Proof Notes

<u>Example</u> $\dfrac{dy}{dx} = \dfrac{2xy}{x^2+y^2}$. If $y = xv$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

And $\dfrac{2xy}{x^2+y^2} = \dfrac{2x^2v}{x^2+x^2v^2} = \dfrac{2v}{1+v^2}$.

So $x\dfrac{dv}{dx} = \dfrac{2v}{1+v^2} - v = \dfrac{2v - v(1+v^2)}{1+v^2}$

So $x\frac{dv}{dx} = \frac{v - v^2}{1+v^2}$

or $\left(\frac{1+v^2}{v-v^2}\right) dv = \frac{dx}{x}$

$$\frac{1+v^2}{v-v^2} = \frac{1}{v} - \frac{1}{v+1} - \frac{1}{v-1}$$

$$\int\left(\frac{1}{v} - \frac{1}{v+1} - \frac{1}{v-1}\right) dv = \ln v - \ln(v+1) - \ln(v-1)$$

$$= \int \frac{dx}{x} = \ln x + c$$

$\therefore$

$$\ln v - (\ln(v+1) + \ln(v-1))$$
$$= \ln v - \ln(v^2-1) = \ln\left(\frac{v}{v^2-1}\right) = \ln x + c$$

or $\frac{v}{v^2-1} = Ax$ $\qquad A = e^c$

After taking exponentials of both sides

Now $y = xv \Rightarrow v = y/x$.

Hence $\frac{y/x}{(y/x)^2-1} = \frac{xy}{y^2-x^2} = Ax$

Or $\frac{y}{y^2-x^2} = A$. This is the implicit solution

<u>Second Order linear Equations</u> These have the form

$$y'' + p(x)y' + q(y) = R(x)$$

The simplest case is when $R = 0$ and p, q are constants

If $y'' + by' + cy = 0$, $b, c \in \mathbb{R}$ then the solutions are exponentials.

ie. $y = Ae^{m_1 x} + Be^{m_2 x}$ is the general form of solution. A, B constants, what are m_1, m_2?

Put $y = e^{\lambda x}$. Then $y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

So $\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x}$

$= e^{\lambda x}(\lambda^2 + b\lambda + c) = 0$.

$\therefore \lambda^2 + b\lambda + c = 0$ This is the auxilliary eqn roots give m_1, m_2

Example

$y'' + 4y' + 3y = 0$. $\lambda^2 + 4\lambda + 3 = 0$

$\Rightarrow (\lambda+3)(\lambda+1) = 0$

$\lambda = -3, -1$ are roots. (ie m_1, m_2)

$y = Ae^{-3x} + Be^{-2x}$