Equations of Order 2

Second order constant coefficient linear equations are straightforward to solve

If $y'' + ay' + by = 0$ we have the auxilliary equation

$\lambda^2 + a\lambda + b = 0$ (*)

There are three possibilities.

(1) (*) has two distinct real roots m_1, m_2.

Gen Solution is $y = Ae^{m_1 x} + Be^{m_2 x}$

(2) (*) has a single repeated root m.

Gen Solution is $y = (Ax + B)e^{mx}$.

(3) Roots are $\alpha \pm i\beta$.

$$y = Ae^{\alpha x + i\beta x} + Be^{\alpha x - i\beta x}$$

$$= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x})$$

$$= e^{\alpha x}\big(A(\cos(\beta x) + i\sin(\beta x)) + B(\cos\beta x - i\sin(\beta x))\big)$$

where $$= e^{\alpha x}(C_1\cos\beta x + C_2\sin(\beta x))$$

$B + A = C_1$, $iA - iB = C_2$.

<u>Example</u> $y'' - 4y' + 13y = 0$

Then $\lambda^2 - 4\lambda + 13 = 0$ is the auxillia equation. So $\lambda = \frac{4 \pm \sqrt{16 - 4.13}}{2} = 2 \pm 3i$

So $y = e^{2x}(C_1\cos(3x) + C_2\sin(3x))$

These are easy. What about $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$?

We will learn in coming weeks how to solve this.

We will assume that our equations have solutions, as detailed in Theorem 2.2. We are interested in finding solutions, not proving their existence.

We need some theory.

First we define

$$Ly = y'' + p(x)y' + q(y)$$

This is a second order linear differential operator. It is linear because

$$L(ay_1 + by_2) = aLy_1 + bLy_2$$

where a, b are constants. If $Ly_1 = Ly_2 = 0$

Then $L(ay_1 + by_2) = 0$

Definition

A collection of functions $\{y_1, \dots, y_n\}$ on an interval I is said to be linearly independent if

$$c_1 y_1 + \dots + c_n y_n = 0 \quad \text{for all}$$

$x \in I$, if and only if $c_1 = c_2 = \dots = c_n = 0$

The idea is that y_1, y_2 say are linearly independent, if

$$y_1(x) \neq k y_2(x)$$

for every $x \in I$ and any k.

eg.

$y_1 = e^x$ $y_2 = x$ are linearly independent on $\mathbb{R}$ as $e^x \neq ax$ all x.

Theorem Suppose that the nth order ODE $y^{(n)}(x) + \dots + a_{n-1}(x)y'(x) + a_n(x)y_n = 0$ has n linearly independent solutions $y_1, \dots, y_n$

Then every solution of the ODE can be written

$$y = \sum_{k=1}^{n} c_k y_k \qquad (+)$$

for some k. $(+)$ is called the <u>general</u> <u>solution</u>.

<u>Corollary</u> An nth order linear ODE can have at most n linearly independent solutions

<u>Question</u> How do we prove that $\{y_1, \ldots, y_n\}$ are linearly independent.

This has a simple answer.

<u>Definition</u> Let $y_1(x)$, $y_2(x)$ be any two solutions of

$$y'' + p(x)y' + q(x)y = 0$$

Then the Wronskian of y_1, y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

More generally

$$W(y_1, \ldots, y_n) = \det \begin{vmatrix} y_1 & \ldots & y_n \\ y_1' & & y_n' \\ & & \\ y_1^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

<u>Theorem</u> $\{y_1, \ldots, y_n\}$ are linearly independent on $I \subseteq \mathbb{R}$, if there is at least one point $x_0 \in I$ such that

$$W(y_1, \ldots, y_n)\Big|_{x=x_0} \neq 0$$

<u>Proof</u> Suppose that $y_1, \ldots, y_n$ are linearly dependent. We can find $a_1, \ldots, a_n \neq 0$ such that

$$a_1 y_1(x) + \ldots + a_n y_n(x) = 0$$

all $x \in I$. Differentiate $n-1$ times

Then
$$a_1 y_1 + \cdots + a_n y_n = 0$$
$$a_1 y_1' + \cdots + a_n y_n' = 0$$
$$\vdots$$
$$\underbrace{a_1 y_1^{(n-1)} + \cdots}_{\overline{W}} + a_n y_n^{(n-1)} = 0$$

$$= \begin{pmatrix} y_1 \ldots & y_n \\ y_1' & y_n' \\ & \\ y_1^{(n-1)} & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} a_1 \\ \\ \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \\ 0 \end{pmatrix}$$

But there are nonzero solutions. So the system does not have a unique solution $\therefore$ $\det \overline{W} = 0$

$\therefore$ linear independence $\Rightarrow \det \overline{W} \neq 0$

The Wronskian is very useful.

<u>Theorem</u> If y_1, y_2 satisfy

$$y'' + p(x) y' + q(x) y = 0$$

Then there is a constant K_{12} such that

$$W(y_1, y_2) = K_{12} e^{-\int p(x)\,dx}$$

<u>Proof</u> Suppose that
$$y_1'' + p(x) y_1' + q(x) y_1 = 0 \quad (A)$$
$$y_2'' + p(x) y_2' + q(x) y_2 = 0 \quad (B)$$

$y_1(B) - y_2(A)$

$$= y_1\left(y_2'' + p(x)y_2' + q(x)y_2\right) - y_2\left(y_1'' + y_1'p(x) + q(x)y_1\right)$$

$$= y_1 y_2'' - y_2 y_1'' + (y_1 y_2' - y_2 y_1')p(x) + (y_1 y_2 - y_2 y_1)q$$

$$= y_1 y_2'' - y_2 y_1'' + W p(x) = 0 \qquad (1)$$

But $\frac{d}{dx}\left(y_1 y_2' - y_2 y_1'\right) = y_1 y_2'' + \cancel{y_1' y_2'} - \cancel{y_2' y_1'} - y_2 y_1''$

So (1) is $\frac{dW}{dx} + p(x)W = 0$

$$\therefore \quad W = K_{12}\, e^{-\int p(x)\,dx}$$

This is useful. Firstly

<u>Theorem</u> If y_1, y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

and $W(y_1, y_2) \neq 0$. Then any solution of (2) can be written

$$y = c_1 y_1 + c_2 y_2$$

<u>Proof</u> Let y_3 be any solution of (2)

Then

$$W(y_3, y_1) = y_3 y_1' - y_3' y_1 = K_{13}\, e^{-\int p(x)\,dx}$$

$$W(y_3, y_2) = y_3 y_2' - y_3' y_2 = K_{23}\, e^{-\int p(x)\,dx}$$

y_1, y_2 are known. So we have a pair of simultaneous equations for y_3. We have

$$\underbrace{y_3\left(y_1 y_2' - y_2 y_1'\right)}_{=K_{12}\, e^{-\int p(x)\,dx}} = K_{23}\, e^{-\int p(x)} y_1(x) - K_{13}\, e^{-\int p(x)} y_2$$

or

$$y_3 = \frac{K_{23}}{K_{12}}\, y_1(x) - \frac{K_{13}}{K_{12}}\, y_2$$

$\therefore\ y_3 = c_1 y_1 + c_2 y_2$

We have another incredibly useful result

<u>Theorem</u> Let y_1 be a nonzero solution of

$$y'' + p(x)y' + q(x)y = 0$$

Then $y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx$

is a second linearly independent solution.

<u>Proof</u> Let $z = \frac{y_2}{y_1}$.

Then $\frac{dz}{dx} = \frac{y_2' y_1 - y_1' y_2}{(y_1)^2}$ by the quotient rule. Thus

$$\frac{d}{dx}\left(\frac{y_2}{y}\right) = \frac{W(y_1, y_2)}{(y_1)^2}$$

$$= \frac{K_{12}\, e^{-\int p(x)dx}}{(y_1)^2}$$

Hence $\frac{y_2}{y_1} = K_{12} \int \frac{e^{-p(x)dx}}{(y_1)^2}\,dx$

Thus $y_2 = K_{12}\, y_1 \times \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx$

But multiplying by $\frac{1}{K_{12}}$ gives another solution

<u>Example</u> Solve $x^2 y'' + 4xy' - 4y = 0$

Now $a(x)y'' + kxy' - ky$

$y = x$ is always a solution

So $y_1 = x$ is a solution of $x^2y'' + 4xy' - 4y = 0$.

First divide by x^2 to put it into the form

$$y'' + p(x)y' + q(x)y = 0$$

Thus $$y'' + \frac{4}{x}y' - \frac{4}{x^2}y = 0$$

$\therefore p(x) = \frac{4}{x}$, $\therefore -\int p(x)\,dx = -\int \frac{4}{x}\,dx = -4\ln x$

$$\therefore e^{-\int p(x)dx} = e^{-4\ln x} = e^{\ln x^{-4}} = \frac{1}{x^4}.$$

So $$y_2 = x\int \frac{e^{-\int p}}{(y_1)^2} = x\int \frac{1}{x^4}\cdot\frac{1}{x^2}\,dx$$

$$= x\int x^{-6}\,dx = -\frac{1}{5}x\left(x^{-5}\right)$$

$$= -\frac{1}{5}x^{-4}.$$

Example Solve $y'' + 2ay' + a^2y = 0$

Aux: $\lambda^2 + 2a\lambda + a^2 = 0$

$(\lambda + a)^2 = 0$ or $\lambda = -a$.

$p(x) = 2a$ $\therefore -\int p(x)\,dx = -2ax$

Hence $$e^{-\int p(x)dx} = e^{-2ax}$$

$$y_2 = y_1\int \frac{e^{-2ax}}{(e^{-ax})^2}\,dx = e^{-ax}\int dx$$

$$= xe^{-ax}$$

Example Solve $u'' + xu' + u = 0$

$u_1 = e^{-x^2/2}$ is a solution

Since $u_1' = -x e^{-\frac{x^2}{2}}$, $u_1'' = (x^2-1) e^{-x^2/2}$

So $u_1'' + x u_1' + u = (x^2 - 1 - x^2 + 1) e^{-\frac{x^2}{2}} = 0$

So $p(x) = x \qquad e^{-\int p(x)\,dx} = e^{-\frac{x^2}{2}}$

Thus $u_2 = u_1 \int \dfrac{e^{-\frac{x^2}{2}}}{(e^{-x^2/2})^2} = e^{-\frac{x^2}{2}} \int e^{\frac{x^2}{2}} dx$

This integral cannot be evaluated in terms of a finite number of elementary functions. So we leave it in this form.