Variation of Parameters We are interested in solving

$$y'' + \bar{p}(x)y' + q(x)y = R(x) \qquad (2.15)$$

Theorem Let y_p be a particular solution of (2.15), i.e. any solution.
Let y_h be the general solution of

$$y'' + \bar{p}(x)y' + q(x)y = 0$$

Then every solution of 2.15 has the form

$$y = y_h + y_p$$

(y_h is a solution of the homogeneous problem which is (2.15) when $R=0$)

Proof Let y be a solution of (2.15)

Set $u = y - y_p$

Then $u'' + p(x)u' + q(x)u = y'' + \bar{p}(x)y' + q(x)y$
$$- \left(y_p'' + \bar{p}(x)y_p' + q(x)y_p\right)$$
$$= R(x) - R(x) = 0$$

So u is a solution of the homogeneous problem. But then u can be written

$$u = c_1 y_1 + c_2 y_2 = y_h$$

where y_1, y_2 are l.i. solutions of

$$y'' + \bar{p}(x)y' + q(x)y = 0$$

$$\therefore \quad c_1 y_1 + c_2 y_2 = y - y_p$$

or $$y = y_p + c_1 y_1 + c_2 y_2 = y_p + y_h$$

Suppose that y_1, y_2 are solutions of

$$y'' + p(x) y' + q(x) y = 0$$

which are nonzero and linearly independent. The idea is to try to find a solution

$$y_p = u(x) y_1(x) + v(x) y_2(x)$$

$$y_p' = u' y_1 + u y_1' + v' y_2 + v y_2'.$$

~~$y_p'' = u'' y_1 + u' y_1' + u y_1'' + u' y_1'$~~

$$y_p'' = u'' y_1 + u' y_1' + u' y_1' + u y_1'' + v'' y_2 + v' y_2' + v' y_2' + v y_2''$$

$$= u'' y_1 + 2u' y_1' + u y_1'' + v'' y_2 + 2v' y_2' + v y_2''$$

Sub into (2.15)

$$u'' y_1 + 2u' y_1' + u y_1'' + v'' y_2 + 2v' y_2' + v y_2''$$

$$+ p(x)(u' y_1 + u y_1' + v' y_2 + v y_2') + q(x)(u y_1 + v y_2)$$

$$= u\underbrace{(y_1'' + p(x) y_1' + q(x) y_1)}_{0} + v\underbrace{(y_2'' + p(x) y_2' + q(x) y_2)}_{0}$$

$$+ u'' y_1 + 2u' y_1' + 2v' y_2' + v'' y_2$$

$$+ p(x)(u' y_1 + v' y_2) = R(x)$$

Assume that $u' y_1 + v' y_2 = 0$

Then $\frac{d}{dx}(u' y_1 + v' y_2) = u'' y_1 + u' y_1' + v'' y_2 + v' y_2' = 0$

So $u'' y_1 + 2(u' y_1' + v' y_2') + v'' y_2$

$$= \underbrace{u'' y_1 + u' y_1' + v' y_2' + v'' y_2}_{0} + u' y_1' + v' y_2' = R(x)$$

So $u'y_1 + v'y_2 = 0$

$u'y_1' + v'y_2' = R(x)$

This is a pair of simultaneous equations for u', v'

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ R \end{pmatrix}$$

Hence $$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\therefore \begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{y_1y_2' - y_2y_1'} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ R \end{pmatrix}$$

$$u' = \frac{-y_2 R}{W(y_1, y_2)}$$

$$v' = \frac{y_1 R}{W(y_1, y_2)}.$$

We integrate to find u, v. We can check that this works. But we will do some examples. This is the method of variation of parameters

<u>Example</u> Solve $y'' + y = \sec x$. $\left(\sec x = \frac{1}{\cos x}\right)$

First solve $y'' + y = 0$

$\lambda^2 + 1 = 0$ is the auxillary equation, $\lambda = \pm i$

So $y_1 = \cos x$, $y_2 = \sin x$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$
$$= 1.$$

Then $u' = \dfrac{-y_2 R}{W} = \dfrac{-\sin x \sec x}{1}$

$$= -\tan x$$

$u = -\int \dfrac{\sin x}{\cos x} dx$ $\quad z = \cos x$
$\quad dz = -\sin x dx$

$$= \int \frac{dz}{z} = \ln z = \ln(\cos x)$$

$$v' = \frac{y_1 R}{W} = \frac{\cos x \sec x}{1}$$
$$= 1.$$

Hence $v = \int 1 dx = x$

Thus
$$y_p = uy_1 + vy_2$$
$$= \cos x \ln(\cos x) + x \sin x$$

General solution is
$$y = c_1 \cos x + c_2 \sin x$$
$$+ \cos x \ln(\cos x) + x \sin x.$$