VOP continued. ①

<u>Example</u> $x^2y'' + 4xy' - 4y = x^2$ (1)

First we solve $x^2y'' + 4xy' - 4y = 0$
On p43 of notes, we found $y_1 = x$, $y_2 = x^{-4}$ are solutions.

Now put (1) into the standard form
So we have

$$y'' + \frac{4}{x}y' - \frac{4}{x^2}y = 1.$$

$\therefore R(x) = 1$. The particular solution is

$$y_p = uy_1 + vy_2. \quad \text{where}$$

$$u' = -\frac{y_2 R}{W}, \quad v' = \frac{y_1 R}{W}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^{-4} \\ 1 & -4x^{-5} \end{vmatrix} = -4x^{-4} - x^{-4} = \frac{-5}{x^4}.$$

So $u' = \dfrac{-x^{-4}}{-5/x^{-4}} = \dfrac{1}{5} \therefore u = \dfrac{1}{5}x$

Next

$$v' = \frac{x}{-5/x^4} = -\frac{1}{5}x^5 \therefore v = -\frac{1}{30}x^6$$

$$\therefore y_p = uy_1 + vy_2 = \frac{1}{5}x \times x - \frac{1}{30}x^6(x^{-4})$$

$$= \left(\frac{1}{5} - \frac{1}{30}\right)x^2 = \frac{1}{6}x^2$$

Thus $y = c_1y_1 + c_2y_2 + \frac{1}{6}x^2$

<u>Example</u> $x^2y'' + xy' - 4y = \ln x$ (2)

First, solve $x^2y'' + xy' - 4y = 0$

Try $y = x^\lambda$. $y' = \lambda x^{\lambda-1}$, $y'' = \lambda(\lambda-1)x^{\lambda-2}$

So $x^2(\lambda(\lambda-1)x^{\lambda-2}) + x\lambda x^{\lambda-1} - 4x^\lambda$

$$= x^\lambda(\lambda^2 - \lambda + \lambda - 4) = 0$$

$$= x^\lambda(\lambda^2 - 4) = 0$$

Cancel x^λ. $\lambda^2 - 4 = 0 \therefore \lambda = \pm 2$

Hence solutions are $y_1 = x^2, y_2 = x^{-2}$.

Now divide (2) by x^2

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}.$$

$R(x) = \frac{\ln x}{x^2}$. Next $W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix}$

$$= x^2(-2x^{-3}) - x^{-2}(2x$$

$$= -\frac{4}{x}.$$

Then $y_p = uy_1 + vy_2$, where

$$u' = \frac{-y_2 R}{W} = \frac{-x^{-2}}{(-4/x)}\frac{\ln x}{x^2} = \frac{1}{4}\frac{\ln x}{x^3}$$

$$u = \frac{1}{4}\int \frac{\ln x}{x^3}dx = \frac{1}{4}\left(-\frac{1}{2}\frac{\ln x}{x^2} + \int \frac{1}{2x^3}dx\right)$$

$$= -\frac{1}{8}\frac{\ln x}{x^2} - \frac{1}{16}x^{-2}$$

$$v' = \frac{y_1 R}{W} = \frac{x^2}{-4/x}\frac{\ln x}{x^2} = -\frac{1}{4}x\ln x$$

$$v = -\frac{1}{4}\int x\ln x\,dx = -\frac{1}{4}\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}dx\right)$$

$$= -\frac{x^2}{8}\ln x + \int \frac{x}{2}\,dx = -\frac{x^2}{8}\ln x + \frac{x^2}{16}$$

$$y_p = uy_1 + vy_2 = x^2\left(\frac{-\ln x}{8x^2} - \frac{1}{16x^2}\right) + \frac{1}{x^2}\left(\frac{x^2}{16} - \frac{x^2\ln x}{8}\right)$$

$$= \frac{-\ln x}{8} - \frac{1}{16} - \frac{\ln x}{8} + \frac{1}{16}$$

$$= \frac{-\ln x}{4}.$$

Given $$y_p = y_1(x)\int_{x_0}^{x} \frac{-y_2(t)R(t)}{W(y_1,y_2)(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)R(t)}{W(y_1,y_2)(t)}\,dt$$

<u>Question</u> How do we actually solve $y'' + p(x)y' + q(x)y = 0$?

There are various methods. We will focus on series methods. The idea is to use a power series.

<u>Example</u> Solve $y' = y$, $y(0) = 1$.

We suppose that $y = \sum_{n=0}^{\infty} a_n x^n$.

Then $$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

Now $$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Hence $$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left((n+1) a_{n+1} - a_n \right) x^n = 0$$

Thus $(n+1) a_{n+1} - a_n = 0$

or $a_{n+1} = \dfrac{a_n}{n+1}$.

$n=0$ $\quad a_1 = \dfrac{a_0}{1} = a_0$

$n=1$ $\quad a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{1 \times 2}$

$n=2$ $\quad a_3 = \dfrac{a_2}{2+1} = \dfrac{a_0}{1 \times 2 \times 3}$

$n=3$ $\quad a_4 = \dfrac{a_3}{4} = \dfrac{a_0}{4!}$

In general $a_n = \dfrac{a_0}{n!}$

So $\quad y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = a_0 e^x$

We have $y(0) = 1 \therefore y(0) = a_0 = 1$

So $y = e^x$

Example Solve $y'' + \omega^2 y = 0$

Let $\quad y = \sum_{n=0}^{\infty} a_n x^n$. $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

So

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \omega^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

But $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

Thus $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n} + \sum_{n=0}^{\infty}\omega^{2}a_{n}x^{n}$

$$= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \omega^{2}a_{n}\right]x^{n} = 0$$

Hence $\quad a_{n+2} = \dfrac{-\omega^{2}a_{n}}{(n+2)(n+1)}, \quad n \geq 0$

$n=0 \qquad a_{2} = \dfrac{-\omega^{2}a_{0}}{2}$

$n=2 \qquad a_{4} = \dfrac{-\omega^{2}a_{2}}{4\times 3} = \dfrac{(-1)^{2}(\omega^{2})^{2}a_{0}}{1\times 2\times 3\times 4}$

$n=4 \qquad a_{6} = \dfrac{-\omega^{2}a_{4}}{6\times 5} = \dfrac{(-1)^{3}(\omega^{2})^{3}a_{0}}{1\times 2\times 3\times 4\times 5\times 6}$

and $\quad a_{8} = \dfrac{(-\omega^{2})^{4}a_{0}}{8!}$

So $\quad a_{2n} = \dfrac{(-\omega^{2})^{n}a_{0}}{(2n)!}$

Now

$n=1 \qquad a_{3} = \dfrac{-\omega^{2}a_{1}}{3\times 2}$

$n=3 \qquad a_{5} = \dfrac{-\omega^{2}a_{3}}{5\times 4} = \dfrac{(-\omega^{2})^{2}a_{1}}{1\times 2\times 3\times 4\times 5}$

$n=5 \qquad a_{7} = \dfrac{-\omega^{2}a_{5}}{7\times 6} = \dfrac{(-\omega^{2})^{3}a_{1}}{7!}$

$$a_{2n+1} = \frac{(-1)^{n}(\omega^{2})^{n}a_{1}}{(2n+1)!}$$

Thus

$$y = \sum_{n=0}^{\infty}a_{n}x^{n} = \sum_{n=0}^{\infty}a_{2n}x^{2n} + \sum_{n=0}^{\infty}a_{2n+1}x^{2n+1}$$

$$= a_{0}\sum_{n=0}^{\infty}\frac{(-1)^{n}\omega^{2n}x^{2n}}{(2n)!} + a_{1}\sum_{n=0}^{\infty}\frac{(-1)^{n}\omega^{2n}x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{(\omega x)^{2n} a_0}{(2n)!} + \frac{a_1}{\omega}\sum_{n=0}^{\infty}(-1)^n \frac{(\omega x)^{2n+1}}{(2n+1)!}$$

$$= a_0 \cos(\omega x) + \frac{a_1}{\omega}\sin(\omega x)$$

We can now solve $y'' + \omega^2 y = R(x)$

What about

$$x^2 y'' + x y' + (x^2 - \alpha^2) y = 0 \quad (3)$$

The solutions are Bessel functions. Named for Friedrich Bessel who was an astronomer. The solutions are given as series. Knowing solutions of (3) allows us to solve many ODE's easily

Example $\quad y'' + x y' + y = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$. $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

So

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 + a_0 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n x^n$$

$$= 2a_2 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1) a_{n+2} + n a_n + a_n\right] x^n$$

So $2a_2 + a_0 = 0$

$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$

$\therefore \; a_{n+2} = \dfrac{-a_n}{n+2}, \quad n \geq 1,$