Equation $y'' + xy' + y = 0$. continued. We found

1

$$a_{n+2} = \frac{-a_n}{n+2}, \quad n \geq 1.$$

We generate the even terms

$n=0$ $\quad a_2 = \frac{-a_0}{2}$

$n=2$, $\quad a_4 = \frac{-a_2}{2+2} = \frac{(-1)^2 a_0}{2\times 4}$

$n=4$, $\quad a_6 = \frac{-a_4}{6} = \frac{(-1)^3 a_0}{2\times 4\times 6}$

$n=6$, $\quad a_8 = \frac{(-1)^4 a_0}{2\times 4\times 6\times 8} = \frac{(-1)^4 a_0}{2^4\, 4!}$

$\ldots$

$$a_{2n} = \frac{(-1)^n a_0}{2^n\, n!}$$

Odd terms

$n=1$ $\quad a_3 = \frac{-a_1}{3}$

$n=3$, $\quad a_5 = \frac{-a_3}{5} = \frac{(-1)^2 a_1}{3\times 5}$

$n=5$, $\quad a_7 = \frac{-a_5}{7} = \frac{(-1)^3 a_1}{3\times 5\times 7}$

$n=7$ $\quad a_9 = \frac{(-1)^4 a_1}{1\times 3\times 5\times 7\times 9}$

$= \frac{(-1)^4\, 2\times 4\times 6\times 8}{9!} a_1$

$= (-1)^4 \frac{2^4\, 4!}{9!} a_1$

$$a_{2n+1} = (-1)^n \frac{2^n\, n!}{(2n+1)!} a_1$$

$$\therefore\ y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n\, n!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n\, n!}{(2n+1)!} x^{2n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{2}\right)^n \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

where $z = -\frac{x^2}{2}$.

Hence $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2^n n!} = e^{-\frac{x^2}{2}}$

What about $\sum_{n=0}^{\infty}(-1)^n \frac{2^n n!\, x^{2n+1}}{(2n+1)!}$?

This does not simplify. to an elementary function.

Let $E(x) = \sum_{n=0}^{\infty}(-1)^n \frac{2^n n!\, x^{2n+1}}{(2n+1)!}$.

Then $y = a_0 e^{-x^2/2} + a_1 E(x)$

When does an ODE of the form
(1) $y'' + p(x)y' + q(x)y = 0$ have power series solutions?

A point $x_0 \in I \subseteq \mathbb{R}$ is an ordinary point for (1) on I, if the functions $p(x)$, $q(x)$ are analytic around x_0.

If every point $x \in I$ is an ordinary point, then every solution of (1) can be written as a power series.

Example Airy's Equation

$$y'' - xy = 0$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n\geq 1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n\geq 2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\therefore \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x \sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=0}^{\infty} a_n x^{n+1}$$

or

$$2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$= 2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$= 2a_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2) a_{n+3} - a_n\right] x^{n+1} = 0$$

$\therefore a_2 = 0$ an $a_{n+3} = \dfrac{a_n}{(n+2)(n+3)}$.

Now

$$a_5 = \frac{a_2}{4 \times 5} = 0$$
$$a_8 = \frac{a_5}{7 \times 6} = 0, \quad a_{11} = 0$$

etc. $a_{3n+2} = 0$ for all n

Now $n = 0$

$$a_3 = \frac{a_0}{2 \times 3}, \quad a_6 = \frac{a_3}{(6 \times 5)} = \frac{a_0}{(2 \times 3)(6 \times 5)}$$

$$a_9 = \frac{1}{(9 \times 8)(6 \times 5)(3 \times 2)} a_0 = \frac{1 \times 4 \times 7}{9!} a_0$$

$$a_{3n} = \frac{1.4 \ldots (3n-5)(3n-2)}{(3n)!} a_0$$

$$a_{3n+1} = \frac{2.5 \ldots (3n-4)(3n-1)}{(3n+1)!} a_1$$

Thus

$$y = a_0 \sum_{n=0}^{\infty} a_{3n} x^{3n} + a_1 \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1}.$$