Airy's Equation continued. This is

$$y'' - xy = 0$$

If $y = \sum_{n=0}^{\infty} a_n x^n$, then we found

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)} \qquad a_2 = a_5 = \cdots = a_{3n+2} = 0$$

We found $a_9 = \frac{1}{9\times 8} \cdot \frac{1}{6\times 5} \cdot \frac{1}{3\times 2} a_0 = \frac{1.4.7}{9!} a_0$

$$a_{12} = \frac{1.4.7.10}{12!}$$

So $\quad a_{3n} = \frac{1.4 \ldots (3n-2)}{(3n)!} a_0, \quad n \geq 1,$

The solution corresponding to a_{3n} is

$$y_0 = a_0\left(1 + \sum_{n=1}^{\infty} \frac{1.4 \ldots (3n-2)}{(3n)!} x^{3n}\right)$$

Similarly $\quad a_{3n+1} = \frac{2.5 \ldots (3n-1)}{(3n+1)!} a_1, \quad n \geq 1$

Then

$$y_1 = a_1\left(x + \sum_{n=1}^{\infty} \frac{2.5 \ldots (3n-1)}{(3n+1)!} x^{3n+1}\right)$$

These are solutions of Airy's equation, but they are not Airy functions.

Definition

$$f(x) = 1 + \sum_{n=1}^{\infty} a_{3n} x^{3n}$$

$$g(x) = x + \sum_{n=1}^{\infty} a_{3n+1} x^{3n+1}$$

Then $Ai(x) = c_1 f(x) - c_2 g(x)$ and

$Bi(x) = \sqrt{3}\,[c_1 f(x) + c_2 g(x)]$.

$C_1 = 3^{-\frac{2}{3}}/\Gamma(\frac{2}{3})$ $\quad C_2 = 3^{-1/3}/\Gamma(1/3)$

$C_1 \simeq 0.355028$, $C_2 \simeq 0.258819$

where $\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$.

Recall that

$\Gamma(n+1) = n!$, $n = 0, 1, 2, 3, \ldots$ These odd constants were chosen so that the Airy functions match certain Bessel functions.

The Airy functions satisfy

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt$$

(To show this we check that $Ai''(x) - x\,Ai(x) = 0$. There is a lot of integration involved)

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[e^{-\frac{1}{3}t^3 + xt} + \sin\left(\frac{1}{3}t^3 + xt\right)\right] dt$$

These are called integral representations

<u>Example</u> Solve $(1-x^2)y'' - 5xy' - 3y = 0$

Notice that this is

$$y'' - \frac{5x}{1-x^2}y' - \frac{3}{1-x^2}y = 0$$

There are singularities at $x = \pm 1$. So we only expect the series solution to converge for $|x| < 1$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. We have

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 5x\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 5n a_n x^n - \sum_{n=0}^{\infty} 3a_n x^n$$

$$= 2a_2 - 3a_0 + 3\times 2a_3 x - 5a_1 x - 3a_1 x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} 5n a_n x^n - \sum_{n=2}^{\infty} 3a_n x^n$$

$$= 2a_2 - 3a_0 + (6a_3 - 8a_1)x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} \Big(n(n-1)a_n + (5n+3)a_n\Big) x^n$$

$$= 2a_2 - 3a_0 + (6a_3 - 8a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2+4n+3)a_n\right]x^n$$

$$= 0$$

$$\therefore \quad a_2 = \frac{3}{2}a_0, \quad a_3 = \frac{8}{6}a_1 = \frac{4}{3}a_1$$

And $\quad a_{n+2} = \dfrac{n^2+4n+3}{(n+2)(n+1)} a_n, \quad n \geq 2$

$$= \frac{(n+3)(n+1)}{(n+2)(n+1)} a_n = \frac{n+3}{n+2} a_n$$

$n=2$, $\quad a_4 = \dfrac{5}{4} a_2 = \dfrac{3\times 5}{2\times 4} a_0$

$$a_6 = \frac{7}{6}a_4 = \frac{3\times 5\times 7}{2\times 4\times 6} a_0$$

$$a_8 = \frac{9}{8}\cdot\frac{7}{6}\cdot\frac{5}{4}\cdot\frac{3}{2}a_0 = \frac{3\times 5\times 7\times 9}{2^4 4!} a_0$$

$$= \frac{2\times 3\times 4\times 5\times 6\times 7\times 8\times 9}{(2^4 4!)^2} a_0 = \frac{9!}{(2^4 4!)^2} a_0$$

So $a_{2n} = \dfrac{(2n+1)!\, a_0}{(2^n n!)^2}$

$n=1$, $a_3 = \frac{4}{3}a_1$, $a_5 = \frac{6}{5}\cdot\frac{4}{3}a_1$, $a_7 = \frac{8}{7}\cdot\frac{6}{5}\cdot\frac{4}{3}a_1$

$$a_9 = \frac{10}{9}\cdot\frac{8}{7}\cdot\frac{6}{5}\cdot\frac{4}{3}a_1 = \frac{1}{2}\cdot\frac{2\times4\times6\times8\times10}{(3\times5\times7\times9)}a_1$$

$$= \frac{1}{2}\,\frac{2^5\,5!}{(3\times5\times7\times9)}a_1$$

$$= \frac{1}{2}\,\frac{2.4.6.8\;2^5\,5!}{9!}a_1$$

$$= \frac{1}{2}\,\frac{2^4\,4!\;2^5\,5!}{9!}a_1 = \frac{2^9\,4!\,5!}{9!}\,\frac{a_1}{2}$$

$$a_{2n+1} = 2^{2n+1}\,\frac{n!\,(n+1)!}{(2n+1)!}\,\tilde{a}_1, \quad \tilde{a}_1 = \frac{a_1}{2}.$$

$$\therefore\; y = a_0 \sum_{n=0}^{\infty} \frac{(2n+1)!}{(2^n n!)^2} x^{2n} + \tilde{a}_1 \sum_{n=0}^{\infty} \frac{2^{2n+1}\,n!\,(n+1)!}{(2n+1)!} x^{2n+1},$$

for $|x|<1$.

There are many many more equations which can be solved by this method, what about equations with singular points at $x=0$?

Regular Singular Points: Method of Frobenius.

There is a problem that arises when $x=0$ is a singularity. This is that a function may not have a Taylor expansion around $x=0$

Frobenius decided to solve DEs by looking for solutions of the form

$$y = x^s \sum_{n=0}^{\infty} a_n x^n, \quad s \text{ is a constant.}$$

$$= a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \cdots$$

So $y' = s a_0 x^{s-1} + (s+1) a_1 x^s + \cdots$

$$= \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$$

and $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$.

<u>Example</u> Solve $2x^2 y'' + xy' - (x+1)y = 0$

Try $y = \sum_{n=0}^{\infty} a_n x^{n+s}$. We have

$$2x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$$

$$- x \sum_{n=0}^{\infty} a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty} (n+s)(n+s-1) 2a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+s+1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= (2s(s-1) + s - 1) a_0 x^s$$

$$+ \sum_{n=1}^{\infty} \left[(n+s)(n+s-1) 2a_n + (n+s) a_n - a_n \right] x^{n+s}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+s+1}$$

If $a_0 \neq 0$, $2s^2 - 2s + s - 1 = 2s^2 - s - 1 = 0$

$\therefore s = 1, \quad s = -1/2$

$$\sum_{n=0}^{\infty} a_n x^{n+s+1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+s}.$$

Thus

$$\sum_{n=1}^{\infty} \left[2(n+s)(n+s-1) + n+s-1\right] a_n x^{n+s}$$

$$- \sum_{n=1}^{\infty} a_{n-1} x^{n+s} = 0$$

$$\sum_{n=1}^{\infty} \left\{(n+s-1)(2n+2s+1)a_n - a_{n-1}\right\} x^{n+s} = 0$$

So $$a_n = \frac{a_{n-1}}{(n+s-1)(2n+2s+1)} \quad n \geq 1.$$

Take $s=1$.

$$a_n = \frac{a_{n-1}}{n(2n+3)}$$

So $$a_1 = \frac{a_0}{1\times 5}, \quad a_2 = \frac{a_0}{1\times 2\times 5\times 7}, \quad a_3 = \frac{a_0}{(1\times 2\times 3)(5\times 7\times 9)}$$

$$a_4 = \frac{a_0}{4!\,(5\times 7\times 9\times 11)}$$

We discover that

$$a_n = 3\,\frac{2^{n+1}(n+1)}{(2n+3)!}\,a_0.$$

The corresponding solution is

$$y = 3a_0 \sum_{n=0}^{\infty} \frac{2^{n+1}(n+1)}{(2n+3)!}\, x^{n+1}.$$