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To recall for $2x^2y'' + xy' - (x+1)y = 0$
Setting,

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

we found $s = 1, -1/2$

$$a_n = \frac{a_{n-1}}{(n+s-1)(2(n+s)+1)}$$

for $s=1$, we got

$$a_n = 3a_0 \frac{2^{n+1}(n+1)}{(2n+3)!}$$

$s = -1/2$ $\quad a_n = \dfrac{a_{n-1}}{(n-3/2)(2n-1+1)} = \dfrac{a_{n-1}}{n(2n-3)}$

$a_1 = \dfrac{a_0}{1(-1)}$, $\quad a_2 = \dfrac{a_1}{2.1} = -\dfrac{a_0}{1\times1\times1\times2}$

$a_3 = \dfrac{a_2}{3\times3} = -\dfrac{a_0}{1\times1\times1\times2\times3\times3} = \dfrac{-a_0}{3!\,1\times3}$

$a_4 = \dfrac{a_3}{4\times5} = \dfrac{-a_0}{(1\times2\times3\times4)(1\times3\times5)}$

$$a_5 = \frac{-a_0}{5!\,(1\times3\times5\times7)} = \frac{-a_0\,(2\times4\times6)}{5!\;7!}$$

$$= -\frac{2^3\,3!}{5!\;7!}\,a_0$$

$$a_n = -\frac{2^{n-2}(n-2)!}{n!\,(2n-3)!} \qquad n>1$$

$$= \frac{2^{n-2}}{n(n-1)(2n-3)!}\,a_0$$

$$y = a_0 x^{-1/2}\left(1 - x - \sum_{n=2}^{\infty} \frac{2^{n-2}\,x^n}{n(n-1)\,(2n-3)!}\right)$$

Consider $$y'' + \frac{P_1(x)}{x-x_0}y' + \frac{P_2(x)}{(x-x_0)^2}y = 0 \quad (1)$$

If P_1, P_2 are analytic around $x = x_0$ then (1) has at least one solution of the form

$$y = (x-x_0)^s \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad (2)$$

x_0 is called a regular singular point

Suppose that the quadratic for s has a single root. Then we only get one solution by this method.

Theorem Let $y'' + p(x)y' + q(x)y = 0$ have a regular singular point at $x=0$. Let $s = s_1$ be the only solution of the quadratic in the method of Frobenius. Let the coefficients be $a_n(s)$ in (2). Then there is a second solution

$$y_2 = y_1(x)\ln x + x^{s_1}\sum_{n=1}^{\infty} a_n'(s_1)\, x^n$$

where y_1 is the solution corresponding to $s = s_1$

Example Find two solutions of

$$x^2y'' - xy' + (1-x)y = 0$$

Put $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, $y' = \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$

$y'' = \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2}$. We have

$$x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} - x\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s} - x\sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s} - \sum_{n=0}^{\infty} (n+s) a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+1}$$

$$= \sum_{n=0}^{\infty} \left[(n+s)(n+s-1) - (n+s) + 1\right] a_o x^{n+s}$$
$$- \sum_{n=0}^{\infty} a_n x^{n+s+1}$$

$$= \underbrace{\left(s(s-1) - s + 1\right)}_{(s^2-2s+1)} a_o x^s + \sum_{n=1}^{\infty} \left[(n+s)(n+s-1) - (n+s) + 1\right] \times a_n x^{n+s}$$
$$- \sum_{n=0}^{\infty} a_n x^{n+s+1}$$

$$(s-1)^2 a_o x^s + (\text{ stuff })$$

Take $s = 1$. Then

$$\sum_{n=1}^{\infty} \left[(n+s)(n+s-1) - (n+s) + 1\right] a_n x^{n+s} - \sum_{n=1}^{\infty} a_{n-1} x^{n+s} = 0$$

$$\therefore \quad a_n = \frac{a_o}{(n+s)(n+s-1) - (n+s) + 1}, \quad n \geq 1$$

If $s = 1$

$$a_n = \left.\frac{a_{n-1}}{(n+s-1)^2}\right|_{s=1} = \frac{a_{n-1}}{n^2}_{\;n\geq 1}$$

$$\therefore \quad a_n = \frac{a_o}{(n!)^2}$$

So $\quad y = a_o x \sum_{n=0}^{\infty} \dfrac{x^n}{(n!)^2}$

For a second solution,

$$a_n(s) = \frac{a_o}{s^2 (s+1)^2 + \cdots (n+s-1)^2}$$

$$\ln a_n(s) = \ln a_0 - \ln(s^2(s+1)^2 \cdots (s+n-1)^2)$$
$$= \ln a_0 - 2\ln s - 2\ln(s+1) - \cdots - 2\ln(n+s-1)$$

Differentiate both sides

$$\frac{d}{ds}\ln(a_n(s)) = \frac{a_n'(s)}{a_n(s)} = -2\left[\frac{1}{s} + \frac{1}{s+1} + \cdots + \frac{1}{n+s-1}\right]$$

$$a_n(1) = \frac{a_0}{(n!)^2}$$

So $$a_n'(1) = \frac{-2a_0}{(n!)^2}\underbrace{\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right]}_{H_n}$$

$$= \frac{-2a_0}{(n!)^2} H_n$$

So $$y = \ln x \sum_{n=0}^{\infty} \frac{a_0 x^{n+1}}{(n!)^2} - 2a_0 \sum_{n=1}^{\infty} \frac{H_n}{(n!)^2} x^{n+1}$$