Bessel's Equation.

We now turn to an extremely important equation.

$$x^2y'' + xy' + (x^2-\alpha^2)y = 0$$

This is Bessel's equation. As before we set

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

So

$$x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + x\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$
$$+ x^2\sum_{n=0}^{\infty} a_n x^{n+s} - \sum_{n=0}^{\infty}\alpha^2 a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s+2} - \sum_{n=0}^{\infty}\alpha^2 a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty}\left\{(n+s)(n+s-1) + n+s - \alpha^2\right\}a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s+2}$$

$$= \left(s(s-1) + s - \alpha^2\right)a_0 x^s + \left((s+1)s + s+1-\alpha^2\right)a_1 x^{s+1}$$
$$+ \sum_{n=2}^{\infty}\left\{(n+s)(n+s-1) + n+s-\alpha^2\right\}a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s+2}$$

$$= (s^2-\alpha^2)a_0 x^s + \left(s^2+2s+1-\alpha^2\right)a_1 x^{s+1}$$
$$+ \sum_{n=2}^{\infty}\left\{(n+s)(n+s-1) + n+s-\alpha^2\right\}a_n x^{n+s}$$
$$+ \sum_{n=2}^{\infty} a_{n-2} x^{n+s}$$

Suppose $a_0 \neq 0$ $s^2 = \alpha^2$. This means $a_1 = 0$

Then $a_n = \dfrac{-a_{n-2}}{(n+s)(n+s-1)+n+s-\alpha^2}$ $n \geq 2$

Put $s = \alpha$.

$$a_n = \frac{-a_{n-2}}{(n+\alpha)(n+\alpha-1)+n+\alpha-\alpha^2}$$
$$= \frac{-a_{n-2}}{(n+\alpha)(n+\alpha-1+1)-\alpha^2}$$
$$= \frac{-a_{n-2}}{(n+\alpha)^2-\alpha^2}$$
$$= \frac{-a_{n-2}}{n^2+2n\alpha+\alpha^2-\alpha^2}$$
$$= \frac{-a_{n-2}}{n(n+2\alpha)}$$

So

$$a_2 = \frac{-a_0}{2(2+2\alpha)} = \frac{-1}{2^2(1+\alpha)}a_0$$
$$a_4 = \frac{-a_2}{4(4+2\alpha)} = \frac{(-1)^2 a_0}{2^4 2!(1+\alpha)(2+\alpha)}$$
$$a_6 = \frac{-a_4}{6(6+2\alpha)} = \frac{(-1)^3 a_0}{2^6 3!(1+\alpha)(2+\alpha)(3+\alpha)}$$

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(1+\alpha)\cdots(n+\alpha)}$$

Recall $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\,dt = \cancel{\left[-t^z e^{-t}\right]_0^\infty}_{0} + z\int_0^\infty t^{z-1}e^{-t}\,dt$$
$$= z\Gamma(z)$$

$$\Gamma(n+\alpha+1) = (n+\alpha)\Gamma(n+\alpha) = (n+\alpha-1)(n+\alpha)\Gamma(n+\alpha-1)$$

$\therefore (1+\alpha)(2+\alpha)\cdots(n+\alpha)\Gamma(1+\alpha) = \Gamma(n+\alpha+1)$

Put $a_0 = \dfrac{1}{2^\alpha \Gamma(1+\alpha)}$. Then

$$a_n = \frac{1}{2^\alpha \Gamma(1+\alpha)(1+\alpha)\cdots(n+\alpha)\, n!\, 2^{2n}}$$

$$= \frac{1}{2^{2n+\alpha}\, n!\, \Gamma(n+\alpha+1)}$$

Definition The Bessel function of the first kind of order α is

$$J_\alpha(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+\alpha}}{2^{2n+\alpha}\, n!\, \Gamma(n+\alpha+1)}$$

If we take $s=-\alpha$ we get exactly $J_{-\alpha}(x)$ and this is also a solution. However, if $\alpha = N$ an integer J_{-N} is a multiple of J_N. Why?

$$J_{-N}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-N}}{2^{2n-N}\, n!\, \Gamma(n-N+1)}$$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$$ γ is Euler's constant

$$\frac{1}{\Gamma(0)} = 0. \quad \frac{1}{\Gamma(-1)} = 0. \quad \frac{1}{\Gamma(-N)} = 0$$ all $N = 0, 1, 2, \ldots$

If $N=2$

$$J_{-2}(x) = 0 + 0 + \sum_{n=2}^{\infty} (-1)^n \frac{x^{2n-2}}{2^{2n-2}\, n!\, \Gamma(n-2+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+2} \frac{x^{2(n+2)-2}}{2^{2(n+2)-2}\, (n+2)!\, \Gamma(n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^{2n+2}\, n!\, \Gamma(n+2+1)} = J_2(x)$$

If α is an integer, then Weber's Bessel function is a second linearly independent solution. It is denoted by $Y_n(x)$. p78 of the notes.

$\alpha \notin \mathbb{Z}$

$$Y_\alpha(x) = \frac{\cos(\pi\alpha) J_\alpha(x) - J_{-\alpha}(x)}{\sin(\pi\alpha)}.$$

The general solution of Bessel's equation is

$$y = c_1 J_\alpha(x) + c_2 Y_\alpha(x)$$

if α is not an integer we can also write

$$y = c_1 J_\alpha(x) + c_2 J_{-\alpha}(x)$$

<u>Modified Bessel functions</u> The equation

$$x^2 y'' + x y' - (x^2 + \alpha^2) y = 0$$

can be solved the same way. One solution is

$$I_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^{2n+\alpha}}{2^{2n+\alpha} n!\, \Gamma(n+\alpha+1)}$$

This is the modified Bessel function of the first kind.

<u>Note</u>

$$J_\alpha(ix) = \frac{(ix)^\alpha}{2^\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{i^{2n} x^{2n}}{2^{2n} n!\, \Gamma(n+\alpha+1)}$$

$$= i^\alpha \sum_{n=0}^{\infty} \frac{x^{2n+\alpha}}{2^{2n} n!\, \Gamma(n+\alpha+1)} = i^\alpha I_\alpha(x)$$

There is a similar problem: $I_1(a) = I_{-1}(x)$ etc.
If n is an integer Macdonald's Bessel function $K_n(x)$ is a linearly independent solution. (p80)

Change of variables. Consider the equation

$$t^2u'' + tu' + (t^2 - \alpha^2)\, u = 0$$

Put $t = ax^r$, $y = x^s u(ax^r)$,
Then we get

$$x^2y'' + \quad -2s)xy' + \left[s^2 - r^2\alpha^2 + a^2r^2x^{2r}\right]y = 0$$

This has solution $y = c_1 x^s J_\alpha(ax^r) + c_2 x^s Y_\alpha(ax^r)$

Example (1) Solve

$$x^2y'' + 7xy' + (4 + 36x^4)\, y = 0$$

Here $1 - 2s = 7$, $s = -3$,
$s^2 - r^2\alpha^2 = 4$, $a^2r^2x^{2r} = 36x^4 \therefore r = 2$,
$4a^2 = 36 \therefore a^2 = 9$
$a = 3$,

$9 - 4\alpha^2 = 4 \Rightarrow 9 - 4 = 4\alpha^2$, $\alpha^2 = 5/4$

$\alpha = \pm\dfrac{\sqrt{5}}{2}$

$$\therefore y = c_1 x^{-3} J_{\frac{\sqrt{5}}{2}}(3x^2) + c_2 x^{-3} J_{-\frac{\sqrt{5}}{2}}(3x^2).$$

Example Solve $y'' + y = 0$ or

$$x^2y'' + x^2y = 0$$

$1 - 2s = 0$
$s = 1/2$, $s^2 - r^2\alpha^2 = 0$, $r^2a^2x^{2r} = x^2$
$\alpha = \pm 1/2$, $a = 1$, $r = 1$

$\therefore\; y = c_1 x^{1/2} J_{1/2}(x) + c_2 x^{\frac{1}{2}} J_{-\frac{1}{2}}(x)$

But we know $y = A\sin x + B\cos x$

In fact

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x,\quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Example $y'' - xy = 0$. This is $x^2y'' - x^3y = 0$

and $1 - 2s = 0$, $s = 1/2$, $s^2 - r^2\alpha^2 = 0$ $\quad a^2r^2x^{2r} = -x^3$

$r = 3/2$

$$\frac{9a^2}{4} = -1,\quad a^2 = -\frac{4}{9} \Rightarrow a = \frac{2}{3}i$$

$$s^2 - r^2\alpha^2 = \frac{1}{4} - \frac{9}{4}\alpha^2 = 0$$

$\therefore\; \alpha^2 = \frac{1}{9}$ $\quad\therefore\; \alpha = \pm\frac{1}{3}$

$$\therefore\; y = C_1 x^{1/2} I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) + C_2 x^{1/2} I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)$$

In fact

$$Ai(x) = \frac{1}{3}\sqrt{x}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right]$$

$$Bi(x) = \sqrt{\frac{x}{3}}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right]$$

(Recall $J_\alpha(ix) = i^\alpha I_\alpha(x)$)

i^α is a constant.