Tutorial Seven Solutions.

Question One.

a) $F(s) = \frac{1}{s+5}$, (b) $F(s) = \frac{d^2}{ds^2}\frac{s}{s^2+4}$, (c) $F(s) = \frac{-3s}{s^2+16} + \frac{15}{(s-1)^2+9}$.

(d) $F(s) = \frac{1}{\sqrt{s^2+a^2}}$, (e) $F(s) = \frac{-(s+a+b)}{(s+a)(s+b)}$,

(f) We have $\cosh(ax) = \frac{1}{2}(e^{ax} + e^{-ax})$ and $\sinh(ax) = \frac{1}{2}(e^{ax} - e^{-ax})$. So

$$\begin{aligned} f(x) &= \sin(ax)\cosh(ax) - \cos(ax)\sinh(ax) \\ &= \frac{1}{2}(e^{ax} + e^{-ax})\sin(ax) - \frac{1}{2}(e^{ax} - e^{-ax})\cos(ax). \end{aligned}$$

Now $\mathcal{L}(e^{ax}f(x)) = F(s-a)$ and $\mathcal{L}(e^{-ax}f(x)) = F(s+a)$. Hence

$$\begin{aligned} \mathcal{L}(f) &= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2} - \frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right] \\ &= \frac{1}{2}\left[\frac{s+2a}{(s-a)^2+a^2} + \frac{s+2a}{(s+a)^2+a^2}\right]. \end{aligned}$$

(g) $f(x) = x^2 \sin x \cos(2x)$ so $\mathcal{L}(f) = \frac{d^2}{ds^2}\mathcal{L}(\sin x \cos(2x))$. Now

$$\begin{aligned} \sin x \cos(2x) &= \frac{1}{2i}(e^{ix} - e^{-ix})\frac{1}{2}(e^{2ix} + e^{-2ix}) \\ &= \frac{1}{4i}(e^{3ix} + e^{-ix} - e^{ix} - e^{-3ix}) \\ &= \frac{1}{2}(\sin(3x) - \sin x). \end{aligned}$$

Thus

$$\mathcal{L}[\sin x \cos(2x)] = \frac{1}{2}\left(\frac{3}{s^2+9} - \frac{1}{s^2+1}\right).$$

Finally

$$\mathcal{L}(f) = \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{3}{s^2+9} - \frac{1}{s^2+1}\right).$$

The derivatives are easy to compute, but I will leave that to you.

(h)

$$\begin{aligned} \int_0^\infty \ln(1+x)e^{-sx}dx &= -\frac{1}{s}\left[\ln(1+x)e^{-sx}\right]_0^\infty + \int_0^\infty \frac{e^{-sx}}{s(1+x)}dx \\ &= \int_0^\infty \frac{e^{-sx}}{s(1+x)}dx. \end{aligned}$$

Put $y = 1+x$ then

$$\int_0^\infty \frac{e^{-sx}}{s(1+x)}dx = \frac{e^s}{s}\int_1^\infty \frac{e^{-sy}}{y}dy.$$

Now put $u = sy$. We have

$$\frac{e^s}{s}\int_1^\infty \frac{e^{-sy}}{y}dy = \frac{e^s}{s}\int_s^\infty \frac{e^{-u}}{u}du.$$

Introduce the Incomplete Gamma function $\Gamma(a,x) = \int_x^\infty u^{a-1}e^{-u}du$. Then we have

$$\mathcal{L}[\ln(1+x)] = \frac{e^s}{s}\Gamma(0,s).$$

Question Two.

By the linearity of the Laplace transform, if

$$aF(s) + bG(s) = a\mathcal{L}(f) + b\mathcal{L}(g)$$

then

$$af(t) + bg(t) = \mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)],$$

so the inverse Laplace transform is linear.

Question Three.

(a) $f(t) = t^2e^{-3t}\cos(4t)$. Hence $\mathcal{L}(f) = \frac{d^2}{ds^2}\left(\frac{s+3}{(s+3)^2+16}\right)$.

(b)

$$\begin{aligned}\sin(2t)\cos t(4t) &= \frac{1}{2i}(e^{2it} - e^{-2it})\frac{1}{2}(e^{4it} + e^{-4it})\\ &= \frac{1}{2}\frac{1}{2i}(e^{6it} + e^{-2it} - e^{2it} - e^{-6it})\\ &= \frac{1}{2}(\sin(6t) - \sin(2t)).\end{aligned}$$

Thus

$$\mathcal{L}[\sin(2t)\cos t(4t)] = \frac{1}{2}\left(\frac{6}{s^2+36} - \frac{2}{s^2+4}\right).$$

(c) $f(t) = e^{-3t}(t+3)^2\sin(3t)\sin(4t)\cos(3t)\cos(4t)$ The point of this question is not the answer so much as the method. The answer is very complicated but it can be produced systematically. The exponential term will shift the value s by 3, so we can ignore it. $(t+3)^2 = t^2 + 6t + 9$ so the answer can be written as a sum of derivatives. So the actual problem is to compute the Laplace transform of $\sin(3t)\sin(4t)\cos(3t)\cos(4t)$. Now by elementary trigonometry

$$\sin(3t)\sin(4t)\cos(3t)\cos(4t) = \frac{1}{4}\sin(6t)\sin(8t).$$

Now

$$\begin{aligned}\sin(6t)\sin(8t) &= \frac{e^{6it} - e^{-6it}}{2i}\frac{e^{8it} - e^{-8it}}{2i}\\ &= -\frac{1}{4}(e^{14it} - e^{-2it} - e^{2it} + e^{-14it})\\ &= -\frac{1}{2}\left(\frac{e^{14it} + e^{-14it}}{2} - \frac{e^{2it} + e^{-2it}}{2}\right)\\ &= \frac{1}{2}(\cos(2t) - \cos(14t)).\end{aligned}$$

Thus this rather complicated problem simplifies a lot. Now we set

$$\begin{aligned} H(s) &= \mathcal{L}[e^{-3t}\frac{1}{2}(\cos(2t) - \cos(14t))] \\ &= \frac{1}{2}\left[\frac{s+3}{(s+3)^2+4} - \frac{s+3}{(s+3)^2+196}\right]. \end{aligned}$$

Hence

$$\mathcal{L}[f](s) = \frac{d^2}{ds^2}H(s) - 6\frac{d}{ds}H(s) + 9H(s).$$

This can be computed explicitly, but I will leave that to any student insane enough to do it. The point is that the apparently complicated problem reduces to something much more manageable.

(d) $f(t) = H(t-3)\sin(3(t-3))\cos(4(t-3))$. The Heaviside function results in the Laplace transform being multiplied by e^{-3s}. So we need the Laplace transform of $\sin(3t)\cos(4t) = \frac{1}{2}(\sin(7t) - \sin t)$. Now

$$\mathcal{L}[\frac{1}{2}(\sin(7t) - \sin t)] = \frac{1}{2}\left(\frac{7}{s^2+49} - \frac{1}{s^2+1}\right).$$

Hence

$$\mathcal{L}[f](s) = \frac{1}{2}e^{-3s}\left(\frac{7}{s^2+49} - \frac{1}{s^2+1}\right).$$

Question Four.

We have

$$J_\alpha(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\alpha}}{2^{2n+\alpha} n!\Gamma(n+\alpha+1)}$$

So

$$\mathcal{L}[J_\alpha(t)] = \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{L}[t^{2n+\alpha}]}{2^{2n+\alpha} n!\Gamma(n+\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2n+\alpha+1)}{2^{2n+\alpha} n!\Gamma(n+\alpha+1)s^{2n+\alpha+1}}.$$

This is all I expect for this question. However this series can be summed, but it takes a lot of work. The summation gives

$$\mathcal{L}[J_\alpha(t)] = \frac{(s+\sqrt{1+s^2})^{-\alpha}}{\sqrt{1+s^2}}.$$

Question Five.

(a) By the convolution theorem $\mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{1}{s}F(s)$, where F is the Laplace transform of f. So

$$\mathcal{L}\int_0^t x^2 e^x dx = \frac{1}{s}\mathcal{L}[x^2e^x] = \frac{1}{s}\frac{d^2}{ds^2}\frac{1}{s-1} = \frac{2}{s(s-1)^3}.$$

(b) By the shifting property

$$\mathcal{L}[e^{2t}\sqrt{t}] = \frac{\Gamma(3/2)}{(s-2)^{3/2}}.$$

(c) $\cos^2 u = \frac{1}{2}(1+\cos(2u))$. So

$$\mathcal{L}[\cos^2 u] = \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+4}.$$

Thus

$$\mathcal{L}\int_0^t \cos^2 u du = \frac{1}{s}\left(\frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+4}\right).$$

Question Six.

(a) $F(s) = \frac{e^{-s}}{(s-2)^2}$, $f(t) = H(t-1)(t-1)e^{2(t-1)}$.

(b) $F(s) = \frac{s}{(s-2)^{3/2}(s^2+1)}$. We have $\mathcal{L}^{-1}\left[\frac{1}{(s-2)^{3/2}}\right] = \frac{2}{\sqrt{\pi}}e^{2t}\sqrt{t}$. So by the convolution theorem

$$\mathcal{L}^{-1}\left[\frac{s}{(s-2)^{3/2}(s^2+1)}\right] = \int_0^t \cos(t-u)\frac{2}{\sqrt{\pi}}e^{2u}\sqrt{u}du,$$

which we will not attempt to evaluate.

(c) $F(s) = e^{-2s}\frac{1}{s^2+(3\pi)^2}$. Then $f(t) = H(t-2)\frac{1}{3\pi}\sin(3\pi(t-2))$.

(d) $F(s) = \frac{s}{s^4(s^2+9)} = \frac{1}{s^4}\frac{s}{s^2+9}$. So by the convolution theorem

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^4}\frac{s}{s^2+9}\right] &= \frac{1}{3!}\int_0^t \cos(3u)(t-u)^3 du \\ &= \frac{1}{729}\cos(3t) - \frac{1}{729} + \frac{1}{162}t^2.\end{aligned}$$

Question Seven. Let $F(s) = \mathcal{L}[f]$.

(a)

$$F(s) = \frac{2s+7}{(s+2)(s^2+9)} = \frac{3}{13}\frac{1}{s+2} + \frac{32}{13}\frac{1}{s^2+9} - \frac{3s}{13(s^2+9)}.$$

Then

$$f(t) = \frac{3}{13}e^{-2t} + \frac{32}{39}\sin(3t) - \frac{3}{13}\cos(3t).$$

(b)

$$F(s) = \frac{s+3}{(s^2+9)(s-4)(s+4)} = \frac{1}{200}\left(\frac{1}{s+4} - \frac{7}{s-4}\right) - \frac{(s+3)}{25(s^2+9)}.$$

Hence

$$f(t) = \frac{1}{200}e^{-4t} - \frac{7}{200}e^{4t} - \frac{1}{25}\cos(3t) - \frac{1}{25}\sin(3t).$$

(c) We have

$$F(s) = \frac{s-9}{(s+3)^2+4} = \frac{s+3-12}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+4} - \frac{12}{(s+3)^2+4}.$$

Now $\mathcal{L}^{-1}[F(s+a)] = e^{-at}f(t)$. Thus the inverse Laplace transform is

$$f(t) = e^{-3t}(\cos(2t) - 6\sin(2t)).$$

(d) We have for $a > 0$

$$F(s) = \frac{1}{s^a}e^{k/s} = \frac{1}{s^a}\left(1 + \frac{k}{s} + \frac{1}{2!}\left(\frac{k}{s}\right)^2 + \frac{1}{3!}\left(\frac{k}{s}\right)^3 + \cdots\right).$$

We invert term by term to obtain

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^a}\right] + \mathcal{L}^{-1}\left[\frac{k}{s^{a+1}}\right] + \frac{1}{2!}\mathcal{L}^{-1}\left[\frac{k^2}{s^{a+2}}\right] + \cdots \\ &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{k^n x^{n+a}}{n!\Gamma(n+a)} \\ &= x^{a-1}\sum_{n=0}^{\infty}\frac{(\sqrt{kx})^{2n}}{n!\Gamma(n+a)} \\ &= x^{a-1}\sum_{n=0}^{\infty}\frac{(2\sqrt{kx})^{2n}}{2^{2n}n!\Gamma(n+a)} \end{aligned}$$

Now

$$I_a(z) = \sum_{n=0}^{\infty}\frac{z^{2n+a}}{2^{2n+a}n!\Gamma(n+a+1)}.$$

Comparing we see that

$$f(x) = \left(\frac{x}{k}\right)^{\frac{a-1}{2}} I_{a-1}(2\sqrt{kx}).$$

Just write out this expression as a series and you will see that it is the same as the one we computed.

Question Eight. This questions combines everything to solve an actual real world problem. The function

$$F(s) = \frac{1}{(1+2st)^{n/2}}\exp\left(-\frac{sx}{1+2st}\right)$$

is the Laplace transform of the transition probability density function for a squared Bessel process. This arises in probability theory and financial modelling. We have

$$\begin{aligned} \frac{1}{(1+2st)^{n/2}}\exp\left(-\frac{sx}{1+2st}\right) &= \frac{1}{(2t)^{n/2}(s+\frac{1}{2t})^{n/2}}\exp\left(-\frac{x(s+\frac{1}{2t}-\frac{1}{2t})}{2t(s+\frac{1}{2t})}\right) \\ &= \frac{1}{(2t)^{n/2}}\exp\left(-\frac{x}{2t}\right)\frac{1}{(s+\frac{1}{2t})^{n/2}}\exp\left(\frac{k}{s+\frac{1}{2t}}\right) \end{aligned}$$

where $k = \frac{x}{(2t)^2}$. Now let us take the inverse Laplace transform variable to be y. We use the fact that

$$\mathcal{L}^{-1}[F(s+a)] = e^{-ay}\mathcal{L}^{-1}[F].$$

Here $a = 1/(2t)$. So

$$\mathcal{L}^{-1}\left[\frac{1}{(1+2st)^{n/2}}\exp\left(-\frac{sx}{1+2st}\right)\right] = \frac{1}{(2t)^{n/2}}\exp\left(-\frac{x+y}{2t}\right)\mathcal{L}^{-1}\left(\frac{1}{s^{n/2}}e^{k/s}\right).$$

We computed this in the last question. The final result after some elementary algebra is that the inverse Laplace transform is

$$p(t,x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{n}{4}-\frac{1}{2}}\exp\left(-\frac{x+y}{2t}\right)I_{\frac{n}{2}-1}\left(\frac{\sqrt{xy}}{t}\right).$$

If you want to know what this all means, that you need to do Advanced Stochastic Processes.