37335 Differential Equations.

Tutorial Ten Solutions.

Question One

From lectures the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{3}\right) e^{-2n^2\pi^2 t},$$

and

$$\begin{aligned}
A_n &= \frac{2}{3}\int_0^3 (2x-1)(x-3)\sin\left(\frac{n\pi x}{3}\right)dx \\
&= \frac{2}{3}\left[\frac{-3(-36+n^2\pi^2(3-7x+2x^2))\cos(\frac{n\pi x}{3}) + 9n\pi(-7+4x)\sin(\frac{n\pi x}{3}))}{n^3\pi^3}\right]_0^3 \\
&= \frac{6(n^2\pi^2+12((-1)^n-1))}{n^3\pi^3}.
\end{aligned}$$

Question Two

From the lecture notes we have that

$$u(x,0) = f(x) = x-3 = \sum_{n=1}^{\infty} A_n \sin(nx),$$

so that

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi}(x-3)\sin(nx)dx \\
&= \frac{-2(\pi-3)(-1)^n-6}{\pi n}.
\end{aligned}$$

We also have

$$u_t(x,0) = x^2-1$$

so

$$\begin{aligned}
B_n &= \frac{2}{n\pi c}\int_0^{\pi}(x^2-1)\sin(nx)dx \\
&= \left[\frac{2\left((2-n^2\left(x^2-1\right))\cos(nx)+2nx\sin(nx)\right)}{\pi c n^4}\right]_0^{\pi} \\
&= -\frac{2\left(n^2+(-1)^n\left(\left(\pi^2-1\right)n^2-2\right)+2\right)}{\pi c n^4}.
\end{aligned}$$

and

$$\begin{aligned}
u(x,t) = &\sum_{n=0}^{\infty}\frac{-2(\pi-3)(-1)^n-6}{\pi n}\sin(nx)\cos(nct) \\
&-\sum_{n=0}^{\infty}\frac{2\left(n^2+(-1)^n\left(\left(\pi^2-1\right)n^2-2\right)+2\right)}{\pi c n^4}\sin(nx)\sin(nct).
\end{aligned}$$

Question Three

The solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4}{n\pi}(-2+(-1)^n)\sin(n\pi x)\cos(n\pi ct)$$

for $n \neq 2$. We also have

$$\begin{aligned} a_2 &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos(x)\, dx \\ &= -\frac{1}{2}. \end{aligned}$$

So that

$$f(x) = -1 + \frac{8}{3}\cos\left(\frac{x}{2}\right) - \frac{1}{2}\cos(x) + \sum_{n=3}^{\infty} \frac{8(-1)^n}{n^2-4}\cos\left(\frac{nx}{2}\right).$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{2\pi} x \sin x \sin\left(\frac{nx}{2}\right) dx \\ &= \frac{16\left((-1)^n - 1\right) n}{\pi\left(n^2-4\right)^2}, \end{aligned}$$

$n \neq 2$. Another integration gives $b_2 = \pi$.

(b).

So we have $f(x) = 1 - \frac{1}{2}x$ for $0 \leq x \leq 1$. Then for the cosine series

$$a_0 = \int_0^1 (1 - \frac{1}{2}x) dx = \frac{3}{4}.$$

$$\begin{aligned} a_n &= 2\int_0^1 (1 - \frac{1}{2}x)\cos(n\pi x) dx \\ &= \frac{(-1)^{n+1}+1}{\pi^2 n^2}. \end{aligned}$$

So

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{\pi^2 n^2}\cos(n\pi x).$$

For the sine series

$$\begin{aligned} b_n &= 2\int_0^1 (1 - \frac{1}{2}x)\sin(n\pi x) dx \\ &= \frac{(-1)^{n+1}+2}{\pi n}. \end{aligned}$$

So

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+2}{\pi n}\sin(n\pi x).$$

Question Five. Now for the odd extensions we have

$$\frac{1}{L}\int_{-L}^{L} |f_o(x)|^2 dx = \frac{2}{L}\int_0^L |f(x)|^2 dx$$

and similarly for the even extension. Thus we have for the function in (a)

$$\pi^2 + \sum_{n \neq 2}^{\infty} \left(\frac{16\left(\left(-1\right)^n - 1\right) n}{\pi \left(n^2 - 4\right)^2} \right)^2 = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin^2 x dx$$
$$= \frac{1}{6}\left(8\pi^2 - 3\right).$$

For the cosine series

$$2 + \frac{64}{9} + \frac{1}{4} + \sum_{n=3}^{\infty} \left(\frac{8(-1)^n}{n^2 - 4} \right)^2 = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin^2 x dx$$
$$= \frac{1}{6}\left(8\pi^2 - 3\right).$$

For the function in (b)

$$\sum_{n=1}^{\infty} b_n^2 = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 = \frac{14}{12}.$$

Question Six.

We have

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{\pi^2}{3},$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx$$
$$= \frac{4(-1)^n}{n^2}.$$

So that

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

Now put $x = 0$. This gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Question Seven.

We know that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. Now we break the sum into the odd and even parts.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^4} &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n)^4}\\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}+\frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4}\\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}+\frac{1}{16}\times\frac{\pi^4}{90}\end{aligned}$$

So that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{90}-\frac{1}{16}\times\frac{\pi^4}{90}=\frac{\pi^4}{96}.$$

Question Eight.

For the sine series

$$\begin{aligned}b_n &= 2\int_0^{\frac{1}{2}}(1-2x)\sin(n\pi x)dx+2\int_{\frac{1}{2}}^{1}\sin(n\pi x)dx\\ &= \frac{2\left(\pi n\left((-1)^{n+1}+\cos\left(\frac{\pi n}{2}\right)+1\right)-2\sin\left(\frac{\pi n}{2}\right)\right)}{\pi^2 n^2}.\end{aligned}$$

For the cosine series

$$\begin{aligned}a_n &= 2\int_0^{\frac{1}{2}}(1-2x)\cos(n\pi x)dx+2\int_{\frac{1}{2}}^{1}\cos(n\pi x)dx\\ &= \frac{-2\pi n\sin\left(\frac{\pi n}{2}\right)-4\cos\left(\frac{\pi n}{2}\right)+4}{\pi^2 n^2}.\end{aligned}$$

and

$$a_0=\int_0^{\frac{1}{2}}(1-2x)dx+\int_{\frac{1}{2}}^{1}dx=\frac{3}{4}.$$

The Fourier series at $x=1/2$ converges to the average of the left and right values. i.e. $1/2(0+1)=1/2$. At $-1/2$ the sine series will converge to the value the Fourier series of the odd extension converges to at that point. This is $-1/2$. The cosine series will converge to $1/2$.

Question Nine.

We use separation of variables as usual. $u(x,y)=X(x)Y(y)$ and $X(0)=X(a)=0$. Now

$$\frac{X''}{X}=-\frac{Y''}{Y}=\lambda.$$

The problem for X is identical to the heat equation. So we have

$$X(x)=A_n\sin\left(\frac{n\pi x}{a}\right)$$

and $\lambda = -\frac{n^2\pi^2}{a^2}$. Hence $Y(y) = Ce^{\frac{n\pi y}{a}} + e^{\frac{-n\pi y}{a}}$. Now u is bounded as $y \to \infty$ so this means that C must equal zero, since $e^{\frac{n\pi y}{a}} \to \infty$ as $y \to \infty$. We then for a solution

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{\frac{-n\pi y}{a}}.$$

Since $u(x,0) = f(x)$

$$A_n = \frac{2}{a}\int_0^a f(z)\sin\left(\frac{n\pi z}{a}\right) dz$$

and

$$u(x,y) = \sum_{n=1}^{\infty}\left(\frac{2}{a}\int_0^a f(z)\sin\left(\frac{n\pi z}{a}\right) dz\right)\sin\left(\frac{n\pi x}{a}\right) e^{\frac{-n\pi y}{a}}.$$

Question Ten.

As before we use separation of variables. We let $u(x,t) = X(x)T(t)$ and we quickly find $X(0) = X(a) = 0$. Then

$$\frac{1}{X}(X'' + kX') = \frac{T'}{T} = \lambda.$$

So that $X'' + kX' - \lambda X = 0$. Solving we find

$$X(x) = c_1 e^{\frac{1}{2}x\left(-\sqrt{k^2+4\lambda}-k\right)} + c_2 e^{\frac{1}{2}x\left(\sqrt{k^2+4\lambda}-k\right)}.$$

We can check that if $k^2 + 4\lambda \geq 0$ then we do not obtain a non zero solution, as in the heat equation. So we put $4\lambda = -k^2 - \frac{n^2\pi^2}{a^2}$. Then we have

$$X(x) = e^{-\frac{kx}{2}}\left(A\cos\left(\frac{n\pi x}{a}\right) + B\sin\left(\frac{n\pi x}{a}\right)\right).$$

The condition that $X(a) = 0$ gives $A = 0$. So the solutions are

$$u_n(x,t) = B_n e^{-\frac{kx}{2}}\sin\left(\frac{n\pi x}{a}\right) e^{-(k^2+\frac{4n^2\pi^2}{a^2})t}.$$

We then set

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{kx}{2}}\sin\left(\frac{n\pi x}{a}\right) e^{-(k^2+\frac{4n^2\pi^2}{a^2})t}.$$

Since $u(x,0) = f(x)$ we have

$$f(x) = \sum_{n=1}^{\infty} B_n e^{-\frac{kx}{2}}\sin\left(\frac{n\pi x}{a}\right)$$

which implies that

$$B_n = \frac{2}{a}\int_0^a e^{\frac{ky}{2}} f(y)\sin\left(\frac{n\pi y}{a}\right) dy.$$

Hence

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{2}{a}\int_0^a e^{\frac{ky}{2}}f(y)\sin\left(\frac{n\pi y}{a}\right)dy\right)e^{-\frac{kx}{2}}\sin\left(\frac{n\pi x}{a}\right)e^{-(k^2+\frac{4n^2\pi^2}{a^2})t}.$$