37335 Differential Equations.

Tutorial Nine Solutions.

Question One.

We have the function $f(x) = x(x+1) = x^2 + x, -\pi < x < \pi$ and $f(x + 2\pi) = f(x)$. Then

the Fourier coefficients are the sum of the coefficients for $f(x) = x^2$ and $g(x) = x$. These were calculated in lectures. So the Fourier series is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2} \cos(nx) + 2\frac{(-1)^{n+1}}{n} \sin(nx) \right).$$

Question Two.

We have $f(x) = 2x^2 - 3x + 2$ for $-1 < x < 1$ with $f(x+2) = f(x)$ for all x. Then

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x)dx = \frac{8}{3},$$

$$a_n = \int_{-1}^{1} (2x^2 - 3x + 2)\cos(n\pi x)dx = \frac{8(-1)^n}{\pi^2 n^2},$$

$$b_n = \int_{-1}^{1} (2x^2 - 3x + 2)\sin(n\pi x)dx = \frac{6(-1)^n}{\pi n}.$$

So

$$f(x) = \frac{8}{3} + \sum_{n=1}^{\infty} \left(\frac{8(-1)^n}{\pi^2 n^2} \cos(n\pi x) + \frac{6(-1)^n}{\pi n} \sin(n\pi x) \right).$$

Question Three.

We expand $f(x) = \sin x$ as a cosine series on the interval $[0, \pi)$. Then

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin x dx = \frac{2}{\pi},$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x dx = \frac{1}{\pi} \int_0^{\pi} \sin(2x)dx$$
$$= \frac{1}{\pi} \left[-\frac{1}{2} \cos(2x) \right]_0^{\pi} = 0,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos(nx)dx = \frac{2}{\pi} \frac{1 + (-1)^n}{1 - n^2}, \; n \neq 1.$$

Thus

$$\sin x = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi}\frac{1+(-1)^n}{1-n^2}\cos(nx),$$

on $[0, \pi)$.

Question Four.

Now we find the sine series for $\cos x$ on $[0, \pi)$. By similar calculations we find $b_1 = 0$ and

$$b_n = \frac{2n(1+(-1)^n)}{\pi(n^2-1)}.$$

Whence

$$\cos x = \sum_{n=2}^{\infty} \frac{2n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx).$$

The function has periodicity 2π

Question Five.

We have $f(x) = 1 - x, 0 \leq x < 1$. The sine coefficients are $b_n = \dfrac{2}{n\pi}$. So

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x).$$

For the cosine series $a_0 = \frac{1}{2}$ and

$$a_n = 2\frac{1-(-1)^n}{\pi^2 n^2}.$$

Hence

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{1-(-1)^n}{\pi^2 n^2}\cos(n\pi x).$$

The periodicity is 2.

Question Six.

We have $f(x) = x - [x]$ on $-2 < x < 2$. Now $f(-x) = -x - [-x] = -x + [x] = -(x - [x]) = -f(x)$. So f is odd. This means that the cosine terms are zero. Between -2 and -1 let $f(x) = Ax + B$. Then $f(-2) = -2A + B == -1$ and $f(-1) = -A + B == 0$. So $f(x) = x + 1$. Similarly between 1 and 2 $f(x) = x - 1$. Since $x - [x] = x$ for $x \in (-1, 1)$

this means

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2}(x-[x])\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{-1}(x+1)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{-1}^{1}(x-[x])\sin\left(\frac{n\pi x}{2}\right)dx \\
&\quad + \frac{1}{2}\int_{1}^{2}(x-1)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{-1}(x+1)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{-1}^{1}x\sin\left(\frac{n\pi x}{2}\right)dx \\
&\quad + \frac{1}{2}\int_{1}^{2}(x-1)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{4}{\pi n}\left((-1)^{n+1}-\cos\left(\frac{\pi n}{2}\right)\right).
\end{aligned}
$$

Thus

$$
x-[x] = \sum_{n=1}^{\infty}\frac{4}{\pi n}\left((-1)^{n+1}-\cos\left(\frac{\pi n}{2}\right)\right)\sin\left(\frac{n\pi x}{2}\right).
$$

Question Seven.

We have $f(x)=x$. So for $n\neq 0$,

$$
\begin{aligned}
\widehat{f}(n) &= \int_0^1 xe^{-2\pi inx}dx \\
&= \frac{i}{2\pi n}.
\end{aligned}
$$

We also have

$$
\widehat{f}(0) = \int_0^1 xdx = \frac{1}{2}.
$$

Whence

$$
x = \frac{1}{2} + \sum_{n\neq 0}\frac{i}{2\pi n}e^{2\pi inx}.
$$

Question Eight.

We have $u_t = \frac{1}{2}u_{xx}$, $0\leq x\leq 1$, $t>0$ with $u_x(0,1)=u_x(1,t)=0$ and $u(x,0)=f(x)$. We let $u(x,t)=X(x)T(t)$. As in the case from lectures we obtain

$$
\frac{X''}{X} = 2\frac{T'}{T} = \lambda.
$$

We also have $X'(0)=X'(1)=0$. Then we suppose $\lambda=k^2>0$. This gives

$$
X(x) = Ae^{kx}+Be^{-kx}.
$$

$X'(0) = kA - kB = 0$, and $X'(1) = kAe^k - kBe^{-k} = 0$ gives $A = B = 0$, so λ cannot be positive. Taking $\lambda = 0$ gives $X(x) = Ax + B$ and $X'(0) = A = 0$. This gives $X = B$. This is not much use. Take $\lambda = -k^2$ to get

$$X(x) = A\cos(kx) + B\sin(kx).$$

We then have

$$X'(0) = -kA\sin(k0) + kB\cos(0) = 0$$

or $B = 0$. The condition $X'(1) = 0$ gives $X'(1) = -kA\sin k = 0$ or $k = n\pi$. Hence $\lambda = -n^2\pi^2$, $n = 0, 1, 2, ...$

$$X(x) = A\cos(n\pi x).$$

Solving for t gives $T(t) = Ce^{-2n^2\pi^2 t}$. So we have solutions

$$u_n(x,t) = A_n\cos(n\pi x)e^{-2n^2\pi^2 t}.$$

We then form a solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n\cos(n\pi x)e^{-2n^2\pi^2 t}.$$

The condition $u(x,0) = f(x)$ gives

$A_n = 2\int_0^1 f(y)\cos(n\pi y)dy$ for $n > 0$ and $A_0 = \int_0^1 f(x)dx$.