37335 Differential Equations.

Tutorial Eight Solutions.

Question One.

(a)

We have

$$x'' + 4x = 5e^{-t}$$
$$x(0) = 2, x'(0) = 3.$$

Take the Laplace transform of both sides to get

$$(s^2+4)X(s) - 2s - 3 = \frac{5}{1+s}.$$

Here $X(t) = \int_0^\infty x(t)e^{-st}dt$. So we see that

$$\begin{aligned} X(s) &= \frac{5}{(1+s)(s^2+4)} + \frac{2+3s}{s^2+4} \\ &= \frac{1}{1+s} + \frac{1}{s^2+4} - \frac{s}{s^2+4} + \frac{2s+3}{s^2+4} \\ &= \frac{1}{1+s} + \frac{4}{s^2+4} + \frac{s}{s^2+4}. \end{aligned}$$

We take the inverse Laplace transform to get

$$x(t) = e^{-t} + \cos(2t) + 2\sin(2t).$$

(b) We solve the initial value problem

$$x'' + 2x' + x = 4\sin t$$
$$x(0) = -2,\ x'(0) = 1.$$

Laplace transforming gives

$$(s^2+2s+1)X(s) - 2x(0) - sx(0) - x'(0) = \frac{4}{s^2+1}.$$

Or

$$\begin{aligned} X(s) &= \frac{4}{(s+1)^2(s^2+1)} + \frac{-4-2s+1}{(s+1)^2} \\ &= \frac{4}{(s+1)^2(s^2+1)} - \frac{2(s+1)+1}{(s+1)^2} \\ &= \frac{4}{(s+1)^2(s^2+1)} - \frac{1}{(s+1)^2} - \frac{2}{s+1} \\ &= \frac{2}{(s+1)^2} + \frac{2}{(s+1)} - \frac{2s}{s^2+1} - \frac{1}{(s+1)^2} - \frac{2}{s+1} \\ &= \frac{1}{(s+1)^2} - \frac{2s}{s^2+1}. \end{aligned}$$

Taking the inverse Laplace transform gives $x(t) = te^{-t} - 2\cos t$.

(c) We solve the problem

$$x'' + 4x' + 5x = 25t,$$

subject to $x(0) = 0, x'(0) = 2$.

We take the Laplace transform of both sides. This gives

$$(s^2 + 4s + 5)X(s) - 2 = \frac{25}{s^2}.$$

So that

$$X(s) = \frac{2}{s^2 + 4s + 5} + \frac{25}{s^2(s^2 + 4s + 5)}$$

Now

$$\frac{25}{s^2(s^2 + 4s + 5)} = \frac{4s + 11}{s^2 + 4s + 5} + \frac{5}{s^2} - \frac{4}{s}.$$

Which gives

$$\begin{aligned} X(s) &= \frac{4s + 13}{s^2 + 4s + 5} + \frac{5}{s^2} - \frac{4}{s} \\ &= \frac{4(s + 2 - 2) + 13}{(s + 2)^2 + 1} + \frac{5}{s^2} - \frac{4}{s} \\ &= \frac{4(s + 2) + 5}{(s + 2)^2 + 1} + \frac{5}{s^2} - \frac{4}{s} \end{aligned}$$

Now we use the result that $\mathcal{L}^{-1}(F(s + a)) = e^{-at}\mathcal{L}^{-1}(F(s))$. So that

$$\mathcal{L}^{-1}\left(\frac{4(s + 2)}{(s + 2)^2 + 1}\right) = e^{-2t}\mathcal{L}^{-1}\left(\frac{4s}{s^2 + 1}\right) = 4e^{-2t}\cos t$$

Similarly

$$\mathcal{L}^{-1}\left(\frac{5}{(s + 2)^2 + 1}\right) = e^{-2t}\mathcal{L}^{-1}\left(\frac{5}{s^2 + 1}\right) = 5e^{-2t}\sin t$$

Since $\mathcal{L}(t) = 1/s^2$ and $\mathcal{L}(1) = 1/s$ we therefore have

$$x(t) = 5e^{-2t}\sin t + 4e^{-2t}\cos t + 5t - 4.$$

Question Two.

To solve the problem

$$x'' + x = f(t), x(0) = x'(0) = 0,$$

where

$$f(t) = \begin{cases} t, & 0 \le t \le 1, \\ 1, & t > 1, \end{cases}$$

we take the Laplace transform. We have

$$\begin{aligned}\int_0^\infty f(t)e^{-st}dt &= \int_0^1 te^{-st} + \int_1^\infty e^{-st}dt \\ &= -\frac{d}{ds}\int_0^1 e^{-st}dt + \left[-\frac{1}{s}e^{-st}\right]_1^\infty \\ &= -\frac{d}{ds}\left[-\frac{1}{s}e^{-st}\right]_0^1 + \frac{e^{-s}}{s} \\ &= -\frac{d}{ds}\frac{1-e^{-s}}{s} + \frac{e^{-s}}{s} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} \\ &= \frac{1-e^{-s}}{s^2}.\end{aligned}$$

So we have

$$(s^2+1)X(s) = \frac{1-e^{-s}}{s^2},$$

or

$$X(s) = \frac{1}{s^2+1}\left(\frac{1-e^{-s}}{s^2}\right).$$

Now by partial fractions

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Thus

$$X(s) = \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)(1-e^{-s})$$

and

$$X(t) = t - \sin t - ((t-1) - \sin(t-1))H(t-1).$$

H is the Heaviside step function.

Question Three.

We have the problem

$$x'' - x = f(t), x(0) = 1, x'(0) = 0,$$

where

$$f(t) = \begin{cases} 0, & 0 \le t \le 1, \\ t-1, & t > 1. \end{cases}$$

Now

$$\int_0^\infty f(t)e*-stdt = \int_1^\infty (t-1)e^{-st}dt$$
$$= \frac{e^{-s}}{s^2}.$$

Thus

$$(s^2-1)X(s) - sx(0) - x'(0) = \frac{e^{-s}}{s^2}.$$

Hence

$$X(s) = \frac{s}{s^2-1} + \frac{e^{-s}}{s^2(s^2-1)}.$$

partial fractions gives $\mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right] = \cosh t.$

$$\frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2}.$$

Thus

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-1} - \frac{1}{s^2}\right] = \sinh t - t.$$

Hence

$$x(t) = \cosh t + H(t-1)(\sinh(t-1) - (t-1)).$$

Question Four.

We obtain a solution to

$$x'' + x = f(t), x(0) = 0, x'(0) = 1,$$

where f possesses a Laplace transform $F(s)$. We see that

$$(s^2+1)X(s) - sx(0) - x'(0) = F(s).$$

Hence

$$X(s) = \frac{1}{s^2+1} + \frac{F(s)}{s^2+1}.$$

By the convolution theorem

$$x(t) = \sin t + \int_0^t f(t-u)\sin u du.$$

Question Five. We have

$$x' + 2x - 2y = 0$$
$$-x + y' + y = 2e^t$$

with $x(0) = 0, y(0) = 1$. We Laplace transform to obtain

$$(s+2)X(s) - 2Y(s) = 0$$
$$-X(s) + (s+1)Y(s) - y(0) = \frac{2}{s-1}.$$

We can write this as

$$\begin{pmatrix} s+2 & -2 \\ -1 & s+1 \end{pmatrix} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{s-1}+1 \end{pmatrix}.$$

So we have

$$\begin{aligned} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} &= \begin{pmatrix} s+2 & -2 \\ -1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{2}{s-1}+1 \end{pmatrix} \\ &= \frac{1}{(s+2)(s+1)-2} \begin{pmatrix} s+1 & 2 \\ 1 & s+2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{2}{s-1}+1 \end{pmatrix} \\ &= \frac{1}{s^2+3s} \begin{pmatrix} s+1 & 2 \\ 1 & s+2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{2}{s-1}+1 \end{pmatrix} \end{aligned}$$

We therefore have

$$\begin{aligned} X(s) &= \frac{2}{s(s+3)} \left(\frac{2}{s-1} + 1 \right), \\ Y(s) &= \frac{s+2}{s(s+3)} \left(\frac{2}{s-1} + 1 \right). \end{aligned}$$

Inverting the Laplace transform we have

$$\begin{aligned} x(t) &= e^t - \frac{1}{3}e^{-t} - \frac{2}{3}, \\ y(t) &= \frac{3}{2}e^t + \frac{1}{6}e^{-3t} - \frac{2}{3}. \end{aligned}$$

Question Six.

We have $u_{xt} + \sin t = 0,\ t > 0$, with $u(x,0) = x$ and $u(0,t) = 0$. Taking the Laplace transform in x to get

$$\frac{\partial}{\partial x} \int_0^\infty u_t e^{-st} dt + \frac{1}{s^1+1} = 0.$$

This is

$$\left(s\frac{\partial \overline{U}}{\partial x} - 1 \right) + \frac{1}{s^2+1} = 0.$$

Hence

$$\overline{U}_x = \frac{1}{s} - \frac{1}{s(s^2+1)}.$$

Integrating gives

$$\overline{U} = x\left(\frac{1}{s} - \frac{1}{s(s^2+1)} \right) + C.$$

Since $u(0,t) = 0$ we have $\overline{U}(0,t) = 0$. This $C = 0$. Now

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

So

$$x\left(\frac{1}{s}-\frac{1}{s(s^2+1)}\right) = \frac{x}{s} - x\left(\frac{1}{s}-\frac{s}{s^2+1}\right)$$
$$= \frac{xs}{s^2+1}.$$

Hence we find on inverting the Laplace transform that $u(x,t) = x\cos t$.

Question Seven.

We solve the wave equation with a source term

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - k\sin(\pi x),\ 0 \le x \le 1,\ t > 0,$$

subject to $u(x,0) = 0, u_t(x,0) = 0, u(0,t) = u(1,t) = 0$. (Note the typo in the question.) Take the Laplace transform in t to obtain

$$\frac{\partial^2 \overline{U}}{\partial x^2} = \frac{s^2}{c^2}\overline{U} - \frac{k}{s}\sin(\pi x).$$

We solve the Homogenous problem to obtain

$$\overline{U}_h(x,s) = A\cosh\left(\frac{sx}{c}\right) + B\sinh\left(\frac{sx}{c}\right).$$

Now we try undetermined coefficients. Let $\bar{U}_p = a\cos(\pi x) + b\sin(\pi x)$ Then

$$\bar{U}_p' = -a\pi\sin(\pi x) + b\pi\cos(\pi x), \bar{U}_p'' = -a\pi^2\cos(\pi x) - b\pi^2\sin(\pi x).$$

$$-a\pi^2\cos(\pi x) - b\pi^2\sin(\pi x) = \frac{s^2}{c^2}(a\cos(\pi x) + b\sin(\pi x)) - \frac{k}{s}\sin(\pi x).$$

Hence $a = 0,\ -b(\pi^2 + \frac{s^2}{c^2}) = -\frac{k}{s}$. Thus

$$b = \frac{c^2 k}{s(c^2\pi^2 + s^2)}.$$

Thus we have a solution of the inhomogenous ODE given by

$$\overline{U}(x,s) = A\cosh\left(\frac{sx}{c}\right) + B\sinh\left(\frac{sx}{c}\right) + \frac{c^2 k}{s(c^2\pi^2+s^2)}\sin(\pi x).$$

The conditions $u(0,t) = 0$ and $u(1,t) = 0$ give $\overline{U}(0,s) = 0$ and $\overline{U}(1,s) = 0$. Now $\overline{U}(0,s) = A$, so $A = 0$. $U(1,s) = B\sinh\left(\frac{s}{c}\right) = 0$. So $B = 0$. Thus

$$\overline{U}(x,s) = \frac{c^2 k}{s(c^2\pi^2+s^2)}\sin(\pi x).$$

We take the inverse Laplace transform to get

$$u(x,t) = \frac{k}{\pi^2}(1-\cos(\pi ct))\sin(\pi x).$$