37335 Differential Equations.

Tutorial Five Solutions.

Series Solutions. Regular Singular Points.

Question One.

(a) $x^2y'' - 5xy' + (3-x)y = 0$. We let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ as usual and then we have

$$y' = \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$
$$y'' = \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2}.$$

Substituting into the DE gives

$$x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2} - 5x\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1} + (3-x)\sum_{n=0}^{\infty}a_nx^{n+s}$$
$$= \sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s} - \sum_{n=0}^{\infty}5(n+s)a_nx^{n+s} + \sum_{n=0}^{\infty}3a_nx^{n+s}$$
$$-\sum_{n=0}^{\infty}a_nx^{n+s+1} = (s(s-1)-5s+3)a_0x^s + \sum_{n=1}^{\infty}(n+s)(n+s-1)a_nx^{n+s}$$
$$-\sum_{n=1}^{\infty}5(n+s)a_nx^{n+s} + \sum_{n=1}^{\infty}3a_nx^{n+s} - \sum_{n=0}^{\infty}a_nx^{n+s+1}$$
$$= (s^2-6s+3)a_0x^s + \sum_{n=1}^{\infty}[(n+s)(n+s-1)-5(n+s)+3]a_nx^{n+s}$$
$$-\sum_{n=1}^{\infty}a_{n-1}x^{n+s} = 0$$

So we have $s^2-6s+3=0$. This gives $s = 3\pm\sqrt{6}$. Let $s_1 = 3+\sqrt{6}$ and $s_2 = 3-\sqrt{6}$. These are not particularly pleasant but it is not difficult to handle them. This is because the recurrence relation will factorise in the same way. Clearly we have

$$((n+s)(n+s-1)-5(n+s)+3)a_n = a_{n-1}, n \geq 1$$

or

$$a_n = \frac{a_{n-1}}{(n+s)(n+s-1)-5(n+s)+3} = \frac{a_{n-1}}{(n+s-s_1)(n+s-s_2)}.$$

If we take $s = s_1$ Then we have

$$a_n = \frac{a_{n-1}}{n(n+2\sqrt{6})}, n \geq 1.$$

The n term will give us an $n!$ in the denominator. We generate terms in the usual manner to obtain

$$a_1 = \frac{a_0}{1(1+2\sqrt{6})}, a_2 = \frac{a_0}{1 \times 2(1+2\sqrt{6})(2+2\sqrt{6})},$$

$$a_3 = \frac{a_0}{1 \times 2 \times 3(1+2\sqrt{6})(2+2\sqrt{6})(3+2\sqrt{6})},$$

etc. We see that the general form is

$$a_n = \frac{a_0}{n!(1+2\sqrt{6})\cdots(n+2\sqrt{6})}.$$

This gives a solution

$$y = a_0 x^{3+\sqrt{6}} \sum_{n=0}^{\infty} \frac{x^n}{n!(1+2\sqrt{6})\cdots(n+2\sqrt{6})}.$$

This is a perfectly acceptable answer. However we can use the Gamma function to simplify it. We recall that $\Gamma(a+1) = a\Gamma(a), Gamma(a+2) = a(a+1)\Gamma(a), ...$ We let $a_0 = \Gamma(1+2\sqrt{6})$. Then

$$(1+2\sqrt{6})\cdots(n+2\sqrt{6})\Gamma(1+2\sqrt{6}) = \Gamma(n+1+2\sqrt{6}).$$

The solution is then

$$y = x^{3+\sqrt{6}} \sum_{n=0}^{\infty} \frac{x^n}{n!\Gamma(n+1+2\sqrt{6})}.$$

You can easily check that taking $s = s_2$ we obtain the second solution

$$y = x^{3-\sqrt{6}} \sum_{n=0}^{\infty} \frac{x^n}{n!\Gamma(n+1-2\sqrt{6})}.$$

(b) For the equation $2x^2y'' + xy' - (2x+1)y = 0$ we have

$$2x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2} + x\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1} - (2x+1)\sum_{n=0}^{\infty}a_nx^{n+s}$$

$$= \sum_{n=0}^{\infty}2(n+s)(n+s-1)a_nx^{n+s} + \sum_{n=0}^{\infty}(n+s)a_nx^{n+s} - \sum_{n=0}^{\infty}2a_nx^{n+s+1}$$

$$-\sum_{n=0}^{\infty}a_nx^{n+s+1} = (2s(s-1)+s-1)a_0x^s + \sum_{n=1}^{\infty}2(n+s)(n+s-1)a_nx^{n+s}$$

$$+\sum_{n=1}^{\infty}(n+s)a_nx^{n+s} - \sum_{n=1}^{\infty}a_nx^{n+s} - \sum_{n=0}^{\infty}2a_nx^{n+s+1}$$

$$= (2s^2-s-1)a_0x^s + \sum_{n=1}^{\infty}[2(n+s)(n+s-1)+(n+s)-1]a_nx^{n+s}$$

$$-\sum_{n=1}^{\infty}2a_{n-1}x^{n+s} = 0$$

So $2s^2 - s - 1 = 0$. This gives $s = 1, s = -1/2$ and

$$a_n = \frac{2a_{n-1}}{2(n+s)(n+s-1)+(n+s)-1}, n \geq 1.$$

If we take $s = 1$ we get

$$a_n = \frac{2a_{n-1}}{n(2n+3)},\ \ n \geq 1.$$

We then generate the coefficients as follows:

$$\begin{aligned} a_1 &= \frac{2a_0}{1(5)}, a_2 = \frac{2^2a_0}{1\times 2(5\times 7)}, a_3 = \frac{2^3a_0}{1\times 2\times 3(5\times 7\times 9)}, \\ a_4 &= \frac{2^4a_0}{4!(5\times 7\times 9\times 11)} = \frac{2^43!\times 2\times 4\times 6\times 8\times 10a_0}{4!11!} \\ &= 3\frac{2^95!a_0}{4!11!}. \end{aligned}$$

We used here $2\times 4\times 6\times 8\times 10 = 2^5(1.2.3.4.5) = 2^55!$.

In general we have

$$a_n = 3\times\frac{2^{2n+1}(n+1)!}{n!(2n+3)!}.$$

This gives the solution

$$y = a_0x\left(1 + 3\times\sum_{n=1}^{\infty}\frac{2^{2n+1}(n+1)!}{n!(2n+3)!}x^n\right).$$

Note that we separated out the $n = 0$ term since our general formula does not hold for $n = 0$, only $n \geq 1$.

For $s = -1/2$ we have

$$a_n = \frac{2a_{n-1}}{n(2n-3)},\ n \geq 1.$$

We iterate as before and we have $a_1 = \frac{2a_0}{1(-1)}$, $a_2 = \frac{2^2 a_0}{1\times 2((-1)(1))}$, $a_3 = \frac{2^3 a_0}{1\times 2\times 3((-1)(1)(3))}$ etc. The corresponding solution can be written

$$y = a_0 x^{-1/2}\left(1 - 2x - \sum_{n=2}^{\infty} \frac{2^{2n-1}(n-1)!x^n}{n!(2n-3)!}\right).$$

(c) The equation $2xy'' + 3y' - xy = 0$ with $\sum_{n=0}^{\infty} a_n x^{n+s}$ becomes

$$\begin{aligned}
&2x\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + 3\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - x\sum_{n=0}^{\infty} a_n x^{n+s} \\
&= \sum_{n=0}^{\infty} 2(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=0}^{\infty} 3(n+s)a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s+1} \\
&= 2a_0 x^{s-1}(2s(s-1)+3s) + 2a_1 x^s(2(s+1)s + 3(s+1)) \\
&\quad + \sum_{n=2}^{\infty}[2(n+s)(n+s-1) + 3(n+s)]a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s+1} \\
&= 2a_0 x^{s-1}(2s(s-1)+3s) + 2a_1 x^s(2(s+1)s + 3(s+1)) \\
&\quad + \sum_{n=2}^{\infty}[2(n+s)(n+s-1) + 3(n+s)]a_n x^{n+s-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+s-1} = 0.
\end{aligned}$$

We will let a_0 be nonzero and set $a_1 = 0$. (You might like to check what happens if you do it the other way round. You should get an answer equivalent to this one). Set $2s^2 + s = 0$. Then $s = 0$ and $s = -1/2$ are the values we need. We also have

$$\begin{aligned}
a_n &= \frac{a_{n-2}}{2(n+s)(n+s-1) + 3(n+s)}, \\
&= \frac{a_{n-2}}{(n+s)(2(n+s-1)+3)}, \\
&= \frac{a_{n-2}}{(n+s)(2n+2s+1)},\ n \geq 2.
\end{aligned}$$

If we take $s_1 = 0$ then we have

$$a_n = \frac{a_{n-2}}{n(2n+1)}, n \geq 2.$$

Notice that if we take $n = 3$ we get $a_3 = \frac{a_1}{3(7)} = 0$. Similarly $a_5 = 0, a_7 = 0$ etc. In general the odd coefficients $a_{2n+1} = 0$ are all zero. Now if we take $n = 2$ we obtain $a_2 = \frac{a_0}{2\times 5}$. For $n = 4$ we have $a_4 = \frac{a_2}{4\times 9} = \frac{a_0}{2\times 4(5\times 9)}$, $a_6 = \frac{a_4}{6\times 13} = \frac{a_0}{2\times 4\times 6(5\times 9\times 13)} = \frac{a_0}{2^3 3!(5\times 9\times 13)}$.

For the even coefficients we have

$$a_{2n} = \frac{a_0}{2^n n!(5 \times \cdots \times (4n+1)}.$$

This holds for $n \geq 0$. So we can write the solution as

$$y = a_0 x^{s_1} \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!(5 \times \cdots \times (4n+1))} = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!(5 \times \cdots \times (4n+1))}.$$

Now we take $s_2 = -1/2$ and we get

$$a_n = \frac{a_{n-2}}{n(2n-1)}, \ n \geq 2.$$

We also find that the odd coefficients a_{2n+1} are all zero. Now take $n = 2$ gives $a_2 = \frac{a_0}{2(3)}$, $a_4 = \frac{a_2}{4 \times 7} = \frac{a_0}{2 \times 4(3 \times 7)}$, then

$$a_6 = \frac{a_0}{2 \times 4 \times 6(3 \times 7 \times 11)} = \frac{a_0}{2^3 3!(3 \times 7 \times 11)}.$$

For $n \geq 1$ we have

$$a_{2n} = \frac{a_0}{2^n n!(3 \times \cdots \times (4n-1))}.$$

So the solution is

$$y = a_0 x^{-1/2} \left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!(3 \times \cdots \times (4n-1))} \right).$$

(d) $(2x^2 - x^3)y'' + (7x - 6x^2)y' + (3 - 6x)y = 0$ is a harder problem. The point is that it is solved the same way. We put $y = \sum_{n=0}^{\infty} a_n x^{n+s}$

and

$$
\begin{aligned}
&(2x^2-x^3)\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2}+(7x-6x^2)\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1}\\
&+(3-6x)\sum_{n=0}^{\infty}a_nx^{n+s}=\sum_{n=0}^{\infty}2(n+s)(n+s-1)a_nx^{n+s}\\
&-\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s+1}+\sum_{n=0}^{\infty}7(n+s)a_nx^{n+s}-\sum_{n=0}^{\infty}6(n+s)a_nx^{n+s+1}\\
&+\sum_{n=0}^{\infty}3a_nx^{n+s}-\sum_{n=0}^{\infty}6a_nx^{n+s+1}\\
&=\sum_{n=0}^{\infty}[2(n+s)(n+s-1)+7(n+s)+3]a_nx^{n+s}\\
&\qquad-\sum_{n=0}^{\infty}[(n+s)(n+s-1)+6(n+s)+6]a_nx^{n+s+1}\\
&=a_0x^s(2s(s-1)+7s+3)+\sum_{n=1}^{\infty}[2(n+s)(n+s-1)+7(n+s)+3]a_nx^{n+s}\\
&\qquad-\sum_{n=0}^{\infty}[(n+s)(n+s-1)+6(n+s)+6]a_nx^{n+s+1}\\
&=a_0x^s(2s^2+5s+3)+\sum_{n=1}^{\infty}[(2(n+s)(n+s-1)+7(n+s)+3)a_n\\
&\quad-((n+s-2)(n+s-1)+6(n+s-1)+6)a_{n-1}]x^{n+s}=0.
\end{aligned}
$$

So we have $2s^2+5s+3=0$ which gives $s=-1$ and $s=-3/2$. We also have

$$a_n=\frac{((n+s-2)(n+s-1)+6(n+s-1)+6)a_{n-1}}{(2(n+s)(n+s-1)+7(n+s)+3)}.$$

Taking $s=-1$ gives

$$a_n=\frac{n(n+1)}{n(2n+1)}a_{n-1}=\frac{n+1}{2n+1}a_{n-1}.$$

So $a_1=\frac{2}{3}a_0$, $a_2=\frac{3}{5}a_1=\frac{2\times3}{3\times5}a_0$, $a_3=\frac{4}{7}a_2=\frac{2\times3\times4}{3\times5\times7}a_0$. We then see that

$$
\begin{aligned}
a_n&=\frac{(n+1)!}{1\times3\times\cdots\times(2n+1)}a_0=\frac{(1\times2\times4\times\cdots2n)(n+1)!}{(2n+1)!}a_0\\
&=\frac{2^nn!(n+1)!}{(2n+1)!}a_0.
\end{aligned}
$$

This gives the solution

$$y = a_0 x^{-1} \sum_{n=0}^{\infty} \frac{2^n n!(n+1)!}{(2n+1)!} x^n.$$

Taking $s = -3/2$ we have after some tedious algebra

$$a_n = \frac{2n+1}{4n} a_{n-1}.$$

Generating terms gives $a_1 = \frac{3}{4}a_0$, $a_2 = \frac{5}{4^2 1\times 2}a_0$, $a_3 = \frac{7}{4^3 3}a_0 = \frac{1\times3\times5\times7}{4^3 3!}a_0$. In general

$$\begin{aligned} a_n &= \frac{(2n+1)!}{4^n(1\times 2\times 4\times\cdots 2n)n!}a_0 \\ &= \frac{(2n+1)!}{2^{3n}(n!)^2}a_0. \end{aligned}$$

So we have a solution

$$y = a_0 x^{-3/2} \sum_{n=0}^{\infty} \frac{(2n+1)!}{2^{3n}(n!)^2} x^n.$$

(e) For $(2x - 2x^2)y'' + (1+x)y' + 2y = 0$ letting $y = \sum_{n=0}^{\infty} a_n x^{n+s}$

$$(2x-2x^2)\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + (1+x)\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$

$$+\,2\sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty} 2(n+s)(n+s-1)a_n x^{n+s-1} - \sum_{n=0}^{\infty} 2(n+s)(n+s-1)a_n x^{n+s}$$

$$+ \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s} + \sum_{n=0}^{\infty} 2a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty}[2(n+s)(n+s-1) + (n+s)]a_n x^{n+s-1}$$

$$- \sum_{n=0}^{\infty}[2(n+s)(n+s-1) - (n+s) - 2]a_n x^{n+s}$$

$$= (2s(s-1)+s)a_0x^{s-1} + \sum_{n=1}^{\infty}[2(n+s)(n+s-1)+(n+s)]a_nx^{n+s-1}$$

$$-\sum_{n=1}^{\infty}[2(n+s-1)(n+s-2)-(n+s-1)-2]a_{n-1}x^{n+s-1}$$

$$= (2s^2-s)a_0x^{s-1} + \sum_{n=1}^{\infty}\left[(2(n+s)(n+s-1)+(n+s))\,a_n\right.$$
$$\left.-\,(2(n+s-1)(n+s-2)-(n+s-1)-2)\,a_{n-1}\right]x^{n+s-1} = 0.$$

Hence $s(2s-1)=0$. So $s=0$ and $s=1/2$ and

$$a_n = \frac{2(n+s-1)(n+s-2)-(n+s-1)-2}{2(n+s)(n+s-1)+(n+s)}a_{n-1}.$$

Taking $s=0$ gives

$$a_n = \frac{2n^2-7n+3}{n(2n-1)}a_{n-1}.$$

It is easy to see that $a_1 = \frac{-2}{1}a_0 = -2a_0$. Then $a_2 = \frac{2(2)^2-14+3}{2(3)}a_1 = -\frac{1}{2}a_1 = a_0$. Now $a_3 = \frac{2(9)-21+3}{3(6-1)}a_2 = 0$. All other terms are zero. So the solution corresponding to $s=0$ is $y = a_0(1-2x+x^2)$.

Taking $s=1/2$ gives

$$a_n = \frac{2n-5}{2n+1}a_{n-1}.$$

We find that $a_1 = \frac{-3}{3}a_0 = -a_0$, $a_2 = \frac{-1}{5}a_1 = \frac{1}{5}a_0$, $a_3 = \frac{1}{7}a_2 = \frac{1}{1\times5\times7}a_0$, $a_4 = \frac{3}{5\times7\times9}a_0$, $a_5 = \frac{3.5}{5.7.9.11}a_0$ It is not hard to see that for $n \geq 2$ we have

$$a_n = \frac{1\times3\times5\cdots\times(n+1)}{3.5.\cdots(2n+1)}a_0.$$

Since this does not hold for the first two terms we separate those terms out in the solution and write

$$y = a_0x^{1/2}\left(1-x+\sum_{n=2}^{\infty}\frac{1\times3\times5\cdots\times(n+1)}{3.5.\cdots(2n+1)}x^n\right).$$

(f) Next we have $x^2(x+2)y''-xy'+(1+x)y=0$. Proceeding as usual gives

$$
\begin{aligned}
&x^2(x+2)\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2} - x\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1}\\
&+(1+x)\sum_{n=0}^{\infty}a_nx^{n+s}\\
&=\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s+1}+\sum_{n=0}^{\infty}2(n+s)(n+s-1)a_nx^{n+s}\\
&-\sum_{n=0}^{\infty}(n+s)a_nx^{n+s}+\sum_{n=0}^{\infty}a_nx^{n+s}+\sum_{n=0}^{\infty}a_nx^{n+s+1}\\
&=\sum_{n=0}^{\infty}[(n+s)(n+s-1)+1]a_nx^{n+s+1}\\
&\qquad+\sum_{n=0}^{\infty}[2(n+s)(n+s-1)-(n+s)+1]a_nx^{n+s}\\
&=(2s(s-1)-s+1)a_0x^s+\sum_{n=0}^{\infty}[(n+s)(n+s-1)+1]a_nx^{n+s+1}\\
&\qquad+\sum_{n=1}^{\infty}[2(n+s)(n+s-1)-(n+s)+1]a_nx^{n+s}\\
&=(2s^2-3s+1)a_0x^s+\sum_{n=1}^{\infty}[(n+s-1)(n+s-2)+1]a_{n-1}x^{n+s}\\
&\qquad+\sum_{n=1}^{\infty}[2(n+s)(n+s-1)-(n+s)+1]a_nx^{n+s}\\
&=(2s^2-3s+1)a_0x^s+\sum_{n=1}^{\infty}([(n+s-1)(n+s-2)+1]a_{n-1}\\
&\quad+[2(n+s)(n+s-1)-(n+s)+1]a_n)x^{n+s}=0.
\end{aligned}
$$

This gives $2s^2-3s+1=0$ and $s=1,1/2$. Finally we have

$$a_n=-\frac{(n+s-1)(n+s-2)+1}{2(n+s)(n+s-1)-(n+s)+1}a_{n-1}.$$

For $s=1$ we obtain

$$a_n=-\frac{n^2-n+1}{n(2n+1)}a_{n-1}.$$

So $a_1=\frac{-1}{3}a_0$, $a_2=-\frac{3}{2(5)}a_1=(-1)^2\frac{1\times1}{1\times2(3\times5)}a_0$,

$$a_3=-\frac{7}{3(7)}a_2=(-1)^3\frac{1\times3\times5}{1\times2\times3(3\times5\times7)}a_0$$

In general

$$\begin{aligned}
a_n &= (-1)^n \frac{(2\times 4\cdots\times 2n)(1\times 1\times 3\times\cdots(n^2-n+1)}{n!(2n+1)!}a_0\\
&= (-1)^n \frac{2^n n!(1\times 1\times 3\times\cdots(n^2-n+1)}{n!(2n+1)!}a_0\\
&= (-1)^n \frac{2^n(1\times 1\times 3\times\cdots(n^2-n+1)}{(2n+1)!}.
\end{aligned}$$

We then have

$$y = a_0 x\sum_{n=0}^{\infty}(-1)^n \frac{2^n(1\times 1\times 3\times\cdots(n^2-n+1)}{(2n+1)!}x^n.$$

For $s = 1/2$ we obtain

$$a_n = \frac{4n^2-8n+7}{4n(2n-1)}a_{n-1}$$

Then $a_1 = \frac{3}{4}a_0$, $a_2 = \frac{3\times 7}{4^2 2!(1\times 3)}a_0$, $a_3 = \frac{3\times 7\times 19}{4^3 3!(1\times 3\times 5)}a_0$ etc.

We find the expression

$$\begin{aligned}
a_n &= \frac{2^{n-1}(n-1)!3\times 7\cdots\times(4n^2-8n+7)}{4^n n!(2n-1)!}\\
&= \frac{1\times 3\times 7\cdots\times(4n^2-8n+7)}{2^{n+1}n(2n-1)!}a_0,\ n\geq 1,
\end{aligned}$$

so that

$$y = a_0 x^{1/2}\left(1+\sum_{n=1}^{\infty}\frac{1\times 3\times 7\cdots\times(4n^2-8n+7)}{2^{n+1}n(2n-1)!}x^n\right).$$

(g) The equation is $3x^2y'' + 8xy' + (x-2)y = 0$.

$$\begin{aligned}
&3x^2\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + 8x\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}\\
&\quad + (x-2)\sum_{n=0}^{\infty}a_n x^{n+s}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}3(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}8(n+s)a_n x^{n+s} + \sum_{n=0}^{\infty}a_n x^{n+s+1}\\
&\quad - \sum_{n=0}^{\infty}2a_n x^{n+s}\\
&= \sum_{n=0}^{\infty}[3(n+s)(n+s-1)+8(n+s)-2]a_n x^{n+s} + \sum_{n=0}^{\infty}a_n x^{n+s+1}
\end{aligned}$$

$$= (3s^2 + 5s - 2)a_0 x^s + \sum_{n=1}^{\infty}[3(n+s)(n+s-1) + 8(n+s) - 2]a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s+1}$$
$$= (3s^2 + 5s - 2)a_0 x^s$$
$$+ \sum_{n=1}^{\infty} \left([3(n+s)(n+s-1) + 8(n+s) - 2]a_n + a_{n-1}\right) x^{n+s} = 0.$$

So $3s^2 + 5s - 2 = 0$ and this gives $s = 1/3$ and $s = -2$. We also have

$$a_n = \frac{-a_{n-1}}{3(n+s)(n+s-1) + 8(n+s) - 2}, \quad n \geq 1.$$

The value $s = -2$ gives

$$a_n = \frac{-a_{n-1}}{n(3n-7)} a_{n-1}.$$

We find after the usual calculations

$$a_n = \frac{(-1)^n}{n!((-4)(-1)(2)(5)\cdots(3n-7)} a_0$$

and

$$y = a_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!((-4)(-1)(2)(5)\cdots(3n-7)} x^n.$$

For $s = 1/3$ we have

$$a_n = \frac{-a_{n-1}}{n(3n+7)} a_{n-1}.$$

The solution is

$$y = a_0 x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(1 \times 10 \times \cdots (3n+7)} x^n.$$

(h) We have $x^2 y'' - x(1+x)y' + y = 0$. So

$$x^2 \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} - x(1+x)\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s}$$
$$= \sum_{n=0}^{\infty}((n+s)(n+s-1) - (n+s) + 1)a_n x^{n+s} - \sum_{n=0}^{\infty}(n+s)a_n x^{n+s+1}$$

$$= (s-1)^2 a_0 x^s$$
$$+\sum_{n=1}^{\infty}[((n+s)(n+s-1)-(n+s)+1)a_n-(n+s-1)a_{n-1}]x^{n+s}=0.$$

So we have $s=1$. And

$$a_n = \frac{n+s-1}{(n+s)(n+s-1)-(n+s)+1}a_{n-1} \tag{0.1}$$
$$= \frac{1}{(n+s-1)}a_{n-1}. \tag{0.2}$$

Clearly taking $s=1$ gives

$$a_n = \frac{1}{n}a_{n-1}.$$

So we immediately have $a_n = \frac{a_0}{n!}$ and

$$y = a_0 x \sum_{n=0}^{\infty}\frac{x^n}{n!} = a_0 x e^x.$$

We can obtain a second solution using this solution in two different ways. We will use the Frobenius method here. Iterating we see that

$$a_n(s) = \frac{a_0}{s(s+1)(s+2)\cdots(n+s-1)}.$$

Take logs to get

$$\ln(a_n(s)) = \ln a_0 - \ln s - \ln(s+1) - \cdots \ln(n+s-1).$$

Differentiate with respect to s to obtain

$$\frac{a_n'(s)}{a_n(s)} = -\frac{1}{s} - \frac{1}{1+s} - \cdots - \frac{1}{n+s-1}.$$

If we take $s=1$ we obtain

$$a_n'(1) = a_n(1)(-\sum k = 1^n \frac{1}{k} = -H_n\frac{a_0}{n!}$$

since $a_n(1) = \frac{a_0}{n!}$ and we have set $H_n = \sum_{k=1}^{n}\frac{1}{k}$. So from the formula in the notes the second solution is

$$y = a_o x e^x - x\sum_{n=1}^{\infty} H_n \frac{a_0}{n!}x^n.$$

Question Two

Solving Differential Equations in Terms of Bessel Functions.

(a) We have $y'' + x^2 y = 0$. Multiply by x^2 to get $x^2y'' + x^4 y = 0$. Compare with the form $x^2y'' + (1-2s)xy' + (s^2 - \alpha^2 r^2 + a^2 r^2 x^{2r})y = 0$. We have $1-2s=0$. So $s=1/2$. Now $a^2r^2x^{2r} = x^4$ giving $2r=4$, so

$r = 2$. $4a^2 = 1$ so $a = 1/2$. Finally $s^2 - \alpha^2 r^2 = \frac{1}{4} - 4\alpha^2 = 0$. Thus $\alpha = \pm\frac{1}{4}$. The solution is therefore

$$y = c_1 x^{\frac{1}{2}} J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) + c_2 x^{\frac{1}{2}} J_{-\frac{1}{4}}\left(\frac{1}{2}x^2\right).$$

We could also write the solution as

$$y = c_1 x^{\frac{1}{2}} J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) + c_2 x^{\frac{1}{2}} Y_{\frac{1}{4}}\left(\frac{1}{2}x^2\right).$$

This is because for non integer values of α the Y_α Bessel functions are just constant multiples of the J_α Bessel functions.

(b) The equation is $x^2y'' + 5xy' + (3 + 4x^2)y = 0$. We have $1 - 2s = 5$, $s = -2$. Next, $x^{2r} = x^2$ so $r = 1$. Then $a^2 = 4$. Hence $a = 2$. Now $s^2 - \alpha^2 r^2 = 4 - \alpha^2 = 3$. So $\alpha = \pm 1$. The solution is then

$$y = c_1 x^{-2} J_1(2x) + c_2 x^{-2} Y_1(2x).$$

(c) For $xy'' - 3y' - 9x^5 y = 0$. Then $x^2y'' - 3xy' - 9x^6 y = 0$. We see that $1 - 2s = -3$ so $s = 2$. Then $x^{2r} = x^6$ or $r = 3$. Then $r^2a^2 = 9a^2 = -9$ or $a = i$. Finally $s^2 - r^2\alpha^2 = 4 - 9\alpha^2 = 0$. So $\alpha = \pm\frac{2}{3}$. Since a is imaginary we use the modified Bessel functions.

$$y = c_1 x^2 I_{\frac{2}{3}}(x^3) + c_2 x^2 I_{-\frac{2}{3}}(x^3).$$

We can also write this as

$$y = c_1 x^2 I_{\frac{2}{3}}(x^3) + c_2 x^2 K_{\frac{2}{3}}(x^3).$$

(d) $x^2y'' + 5xy' + \left(8 + \frac{4}{x^4}\right) y = 0$. Here $1 - 2s = 5$, or $s = -2$. Now $x^{2r} = x^{-4}$ or $r = -2$. Then $4a^2 = 4$ or $a = 1$. Then $s^2 - r^2\alpha^2 = 4 - 4\alpha^2 = 8$, so $4\alpha^2 = -4$. Or $\alpha = \pm i$. The solution is thus

$$y = c_1 x^{-2} J_i\left(\frac{1}{x^2}\right) + c_2 x^{-2} J_{-i}\left(\frac{1}{x^2}\right).$$

Question Three. In Question One, equations (a), (b) and (c) and (h) can be solved in terms of Bessel functions. The solutions are respectively

$$\begin{aligned}
y &= c_1 x I_0\left(\frac{2\sqrt{x}}{\sqrt{3}}\right) + c_1 x K_0\left(\frac{2\sqrt{x}}{\sqrt{3}}\right), \\
y &= c_1 x^{1/4} I_{\sqrt{3}}(\sqrt{2x}) + c_2 x^{1/4} I_{-\sqrt{3}}(\sqrt{2x}), \\
y &= c_1 x^{-1/4} J_{\frac{1}{4}}\left(\frac{x}{\sqrt{2}}\right) + c_2 x^{-1/4} J_{-\frac{1}{4}}\left(\frac{x}{\sqrt{2}}\right) \\
y &= c_1 x^{-1/6} J_{\frac{7}{3}}\left(\frac{2\sqrt{x}}{\sqrt{3}}\right) + c_2 x^{-1/6} J_{-\frac{7}{3}}\left(\frac{2\sqrt{x}}{\sqrt{3}}\right).
\end{aligned}$$

Question Four. The equation $xy'' + (1-x)y' + ny = 0$ is actually a form of the confluent hypergeometric equation which we will encounter later. Since there is an n in the equation we put $y = \sum_{k=0}^{\infty} a_k x^{k+s}$. Then we require

$$\begin{aligned}
&x\sum_{k=0}^{\infty}(k+s)(k+s-1)a_k x^{k+s-2} + (1-x)\sum_{k=0}^{\infty}(k+s)a_k x^{k+s-1} \\
&\quad + n\sum_{k=0}^{\infty} a_k x^{k+s} \\
&= \sum_{k=0}^{\infty}(k+s)(k+s-1)a_k x^{k+s-1} + \sum_{k=0}^{\infty}(k+s)a_k x^{k+s-1} \\
&- \sum_{k=0}^{\infty}(k+s)a_k x^{k+s} + \sum_{k=0}^{\infty} n a_k x^{k+s} \\
&= a_0 x^{s-1}s^2 + \sum_{k=1}^{\infty}[(k+s)(k+s-1) + (k+s)]a_k x^{k+s-1} \\
&\quad - \sum_{k=0}^{\infty}(k+s-n)a_k x^{k+s} \\
&= a_0 x^{s-1}s^2 + \sum_{k=1}^{\infty}[(k+s)((k+s-1)+1)]a_k x^{k+s-1} \\
&\quad - \sum_{k=0}^{\infty}(k+s-n)a_k x^{k+s} \\
&= a_0 x^{s-1}s^2 + \sum_{k=1}^{\infty}(k+s)^2 a_k x^{k+s-1} - \sum_{k=0}^{\infty}(k+s-n)a_k x^{k+s} \\
&= a_0 x^{s-1}s^2 + \sum_{k=0}^{\infty}(k+s+1)^2 a_{k+1} x^{k+s} - \sum_{k=0}^{\infty}(k+s-n)a_k x^{k+s} \\
&= a_0 x^{s-1}s^2 + \sum_{k=0}^{\infty}[(k+s+1)^2 a_{k+1} x^{k+s} - (k+s-n)a_k]x^{k+s} = 0.
\end{aligned}$$

This gives $s = 0$. We also have

$$a_{k+1} = \frac{k+s-n}{(k+s+1)^2}a_k.$$

Taking $s = 0$ we have

$$a_{k+1} = \frac{k-n}{(k+1)^2}a_k,\ k \geq 0.$$

Assume that n is a positive integer. Taking $k = 0$ gives $a_1 = -na_0$. $k = 1$ gives $a_2 = \frac{-n(1-n)a_0}{1^2 \times 2^2}$, $a_3 = \frac{-n(1-n)(2-n)a_0}{(3!)^2}$ etc. Notice that if $k = n$

then

$$a_{n+1} = \frac{(n-n)a_n}{(n+1)^2} = 0.$$

It will then follow that $a_{n+2} = 0$, $a_{n+3} = 0$ etc. So only finitely many term of the solution are nonzero. In other words if n is a positive integer one of the solutions is a polynomial. In general

$$a_k = \frac{-n(1-n)\cdots(k-1-n)a_0}{(k!)^2}, \ k = 1, ..., n,$$

For n a positive integer all other terms are zero. If n is not a positive integer, this formula holds for all k. In fact there is a simple formula for the Laguerre polynomials. We can check that polynomial solutions have the form

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k x^k}{k!},$$

where $\binom{n}{k}$ are the Binomial coefficients. There is a remarkable formula which can be proved by expanding both sides as a Taylor series. For all $|t| < 1$

$$\frac{1}{1-t} \exp\left(-\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} t^n L_n(x).$$

The Laguerre polynomials can also be generated by the Rodriguez formula

$$L_n(x) = (-1)^n e^x \frac{1}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^n\right).$$

For a second solution we can write the coefficients as

$$a_k(s) = \frac{k+s-1-n}{(k+s)^2} a_{k-1}.$$

We only treat the case when n is not a positive integer. If we treat a_0 as a constant and iterate we have

$$a_k(s) = \frac{(s-n)}{(s+1)^2} \frac{(s-n+1)}{(s+2)^2} \cdots \frac{(k+s-1-n)}{(s+k)^2} a_0$$

Taking the natural logarithm gives

$$\ln a_k(s) = \sum_{j=1}^{k} \ln(s-n+j-1) - 2\sum_{j=1}^{k} \ln(s+j) + \ln a_0$$

Differentiating both sides gives

$$\frac{a_k'(s)}{a_k(s)} = \sum_{j=1}^{k} \frac{1}{s-n+j-1} - \sum_{j=1}^{k} \frac{2}{s+j}.$$

Hence

$$\begin{aligned}a_k'(0) &= a_k(0)\left(\sum_{j=1}^{k}\frac{1}{j-1-n} - \sum_{j=1}^{k}\frac{2}{j}\right)\\ &= a_k(0)\left(\sum_{j=1}^{k}\frac{1}{j-1-n} - 2H_k\right)\end{aligned}$$

where $H_k = \sum_{j=1}^{k}\frac{1}{j}$ is the kth harmonic number and

$$a_k(0) = \frac{-n(1-n)\cdots(k-1-n)a_0}{(k!)^2}.$$

If $y_0(x)$ is the solution we found by the method of Frobenius corresponding to $s=0$ we now have a second solution

$$y(x) = y_0(x)\ln x + \sum_{k=1}^{\infty} a_k'(0)x^k.$$

If n is an integer we can construct a second solution, but must be careful not to divide by zero. We can also use the formula for a second solution in the notes. We will return to the problem of finding a second solution to an equation of this form later.

Question Five. We will do this in the workshop in the last week of class. The solution will be posted then. However it would be useful to try and solve it yourself.