37335 Differential Equations.

Tutorial Four Solutions.

Series Solutions. Regular Singular Points.

Question One.
(a). We have to solve $2y'' - xy' - 2y = 0$. This has only ordinary points because the coefficients are analytic everywhere. We let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into the equation gives us

$$2\sum_{n=1}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=2}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n.$$

We look at the first and last series and see that they both start with a term involving x^0. The middle term starts with a term multiplying x. So we take a term out of the first and last series so that all the series start with the *same* power of x.

$$= \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n$$
$$= 4a_2 - 2a_0 + \sum_{n=3}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 2a_n x^n.$$

We write under the same summation sign. So the first series has to be lowered to start at $n = 1$. This means that the n inside the sum has to go up by 2 to compensate.

$$= 4a_2 - 2a_0 + \sum_{n=3}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 2a_n x^n$$
$$= 4a_2 - 2a_0 + \sum_{n=1}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 2a_n x^n$$
$$= 4a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[2(n+2)(n+1)a_{n+2} - (n+2)a_n\right] x^n = 0.$$

So we must have $4a_2 - 2a_0 = 0$ and $2(n+2)(n+1)a_{n+2} - (n+2)a_n = 0$. Hence $a_2 = \frac{1}{2}a_0$ and

$$a_{n+2} = \frac{a_n}{2(n+1)}, n \geq 1.$$

Now let us generate the coefficients. Start with the even values of n. We have $a_2 = \frac{1}{2}a_0$, $a_4 = \frac{a_2}{2(3)} = \frac{a_0}{2^2 \times 3}$. $a_6 = \frac{a_4}{2(5)} = \frac{a_0}{2^3(3\times 5)}$,

$$a_8 = \frac{a_0}{2^4(3 \times 5 \times 7)} = \frac{2 \times 4 \times 6 a_0}{2^4 7!} = \frac{2^3 3! a_0}{2^4 7!}.$$

From this we deduce the general formula

$$a_{2n} = \frac{2^{n-1}(n-1)! a_0}{2^n (2n-1)!}, \quad n \geq 1.$$

Note this formula obviously does not work for $n = 0$, so we treat this as a special case.

Now we do the odd values of n.

$a_3 = \frac{a_1}{2(2)}$, $a_5 = \frac{a_3}{2(4)} = \frac{a_1}{2^2(2\times 4)}$,

$$a_7 = \frac{a_5}{2(6)} = \frac{a_1}{2^3(2 \times 4 \times 6)} = \frac{a_1}{2^6 3!}.$$

We deduce the general formula

$$a_{2n+1} = \frac{a_1}{2^{2n} n!}, n \geq 0.$$

We have a solution which is written

$$y = \sum_{n=0}^{\infty} \frac{a_1}{2^{2n} n!} x^{2n+1} + a_0 \left(1 + \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)!}{2^n (2n-1)!} x^{2n} \right).$$

(b) We have $y'' - (x+1)y = 0$. Note the similarity to the Airy equation. Let $z = x+1$, set $y(x) = Y(z)$. Then $y'' = \frac{d^2 Y}{dz^2}$ so the equation becomes $Y'' - zY = 0$ and this is in the notes.

(c) $y'' - x^2y' - 2xy = 0$ which becomes

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - x^2\sum_{n=1}^{\infty} na_nx^{n-1} - 2x\sum_{n=0}^{\infty} a_nx^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=1}^{\infty} na_nx^{n+1} - \sum_{n=0}^{\infty} 2a_nx^{n+1}$$

$$= 2a_2 + (6a_3 - 2a_0)x + \sum_{n=4}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=1}^{\infty} na_nx^{n+1} - \sum_{n=1}^{\infty} 2a_nx^{n+1}$$

$$= 2a_2 + (6a_3 - 2a_0)x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n$$

$$- \sum_{n=2}^{\infty} 2a_{n-1}x^n$$

$$= 2a_2 + (6a_3 - 2a_0)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n-1+2)a_{n-1}\right] x^n = 0.$$

So we have $a_2 = 0$ and $a_3 = \frac{1}{3}a_0$ and

$$a_{n+2} = \frac{a_{n-1}}{(n+2)}, \ \ n \geq 2.$$

We generate the coefficients in the usual way. So we let $n = 2$, which gives $a_4 = \frac{a_1}{4}$, $a_6 = \frac{a_3}{6} = \frac{1}{3}\frac{1}{6}a_0$, $a_8 = \frac{a_5}{8}$, etc. Now $a_5 = \frac{a_2}{5} = 0$. So $a_8 = 0$. $a_{10} = \frac{a_7}{10}$ and $a_7 = \frac{a_4}{7} = \frac{1}{7}\frac{1}{4}a_1$. The solution can be written

$$y = a_0(1 + \frac{1}{3}x^2 + \frac{1}{18}x^6 + \cdots) + a_1(x + \frac{1}{4}x^4 + \frac{1}{28}x^7 + \cdots).$$

(d) The problem is to solve $(1+x)y'' - y = 0$. $x = 0$ is an ordinary point and $x = -1$ is a singular point. We expand around the ordinary point. So we set $y = \sum_{n=0}^{\infty} a_nx^n$. We find

$$(1+x)\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=0}^{\infty} a_nx^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n$$

$$= 2a_2 - a_0 + \sum_{n=3}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_nx^{n-1} - \sum_{n=1}^{\infty} a_nx^n$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty}(n+1)(n+2)a_nx^n + \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} a_nx^n$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_n + n(n+1)a_{n+1}x^n - a_n\right] x^n = 0.$$

So $a_2 = \frac{1}{2}a_0$ and

$$a_{n+2} = \frac{-n}{n+2}a_{n+1} + \frac{a_n}{(n+2)(n+1)}.$$

We cannot find a closed form expression here, but we can generate as many coefficients as we wish.

$$a_3 = -\frac{1}{3}a_2 + \frac{a_1}{3.2} = -\frac{1}{6}a_0 + \frac{1}{6}a_1,$$

$$a_4 = -\frac{2}{4}a_3 + \frac{1}{3.4}a_2 = -\frac{1}{2}(-\frac{1}{6}a_0 + \frac{1}{6}a_1) + \frac{1}{2.3.4}a_0 = \frac{1}{8}a_0 - \frac{1}{12}a_1.$$

Thus the solution is

$$y = a_0(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots) + a_1(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots).$$

Note: This is how solutions are usually written, since it is generally not possibly to find a closed form for the coefficients a_n.

(e) $y'' - (\sin x)y = 0$. This is a more complicated example. It illustrates what happens when the equation isn't 'nice.' We can still generate a solution. The key is to put $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \sum_{n=0}^{\infty} a_n x^n = 0.$$

The problem is the second term. So we expand it out.

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \sum_{n=0}^{\infty} a_n x^n = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \right) \times$$

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a + 7x^7 + \cdots)$$

$$= \sum_{n=1}^{\infty} b_n x^n.$$

We see that $b_1 = a_0, b_2 = a_1, b_3 = a_2 - \frac{a_0}{3!}$ $b_4 = a_3 - \frac{a_1}{3!}$ etc. In general we have

$$b_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{a_{n-2k}}{(2k+1)!}.$$

The equation is then

$$2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} b_n x^n$$

$$= 2a_2 + \sum_{n=1}^{\infty} n + 2(n+1)a_{n+2}x - \sum_{n=1}^{\infty} b_n x^n = 0.$$

So $a_2 = 0$, and

$$a_{n+2} = \frac{1}{(n+2)(n+1)} b_n.$$

From this we can generate the coefficients. This is quite laborious. However the solution can be written

$$y = a_0 \left(1 + \frac{1}{6}x^3 - \frac{1}{120}x^6 - \frac{1}{120}x^5 + \frac{1}{180}x^6 + \frac{1}{5040}x^7 + \cdots \right) + a_1 \left(x + \frac{1}{12}x^4 - \frac{1}{180}x^6 + \frac{1}{504}x^7 + \cdots \right).$$

(f) The equation is $y'' - xy' - x^2y = 0$. Using the standard formula for a series solution this becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

We take two terms out of the first series and one out of the second so that they all start with a term involving x^2.

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$= 2a_2 + (6a_3 - a_1)x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$= 2a_2 + (6a_3 - a_1)x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} na_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n$$
$$= 2a_2 + (6a_3 - a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n - a_{n-2} \right] x^n = 0.$$

This is a two term recurrence relation. These are typical of what arises in practice, unless the equation has a nice structure. We have $a_2 = 0, a_3 = \frac{1}{6}a_1$ and

$$a_{n+2} = \frac{na_n + a_{n-2}}{(n+1)(n+2)}, \ n \geq 2.$$

Then $a_4 = \frac{2a_2+a_0}{3.4} = \frac{a_0}{3.4}$, $a_5 = \frac{3a_3+a_1}{4.5} = (\frac{1}{2.4.5} + \frac{1}{4.5})a_1 = \frac{3a_1}{2.4.5}$, $a_6 = \frac{4a_4+a_2}{5.6} = \frac{4}{5.6}a_4 = \frac{4a_0}{3.4.5.6}a_0$ etc. So the solution is

$$y = a_0(1 + \frac{1}{12}x^4 + \frac{1}{90}x^6 + \cdots) + a_1(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots).$$

For two term recurrence relations, it is rarely possible to find a closed form for the coefficients. However with a computer package we can generate thousands of coefficients in a few seconds, so we can obtain a solution to any degree of accuracy we desire. In practice the coefficients are obtained by a computer package.

(g) We now have $y'' - (x^2+1)y = 0$. This leads to

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 - a_0 + (6a_3 - a_1)x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=2}^{\infty} a_n x^n$$

$$= 2a_2 - a_0 + (6a_3 - a_1)x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n - \sum_{n=2}^{\infty} a_n x^n$$

$$= 2a_2 - a_0 + (6a_3 - a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-2} - a_n\right] x^n$$

So $a_2 = \frac{1}{2}a_0$, $a_3 = \frac{1}{6}a_1$,

$$a_{n+2} = \frac{a_n + a_{n-2}}{(n+1)(n+2)}, \quad n \geq 2.$$

We generate the coefficients in the usual manner. We get two solutions

$$y = a_0(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8 + \cdots)$$

and

$$y = a_1(x + \frac{1}{6}x^3 + \frac{7}{120}x^5 + \cdots).$$

In fact the equation can be transformed into the equation

$$y'' - \left(\frac{x^2}{4} + a\right) y = 0.$$

This has solutions that are called parabolic cylinder functions. There are two linearly independent solutions which can be written

$$y_1 = e^{-\frac{1}{4}x^2}\,{}_1F_1\left(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}x^2\right)$$
$$y_1 = xe^{-\frac{1}{4}x^2}\,{}_1F_1\left(\frac{1}{2}a + \frac{3}{4}, \frac{3}{2}, \frac{1}{2}x^2\right).$$

The function ${}_1F_1(m,n,z)$ is a confluent hypergeometric function which we will see later.

(h) $(x^2+1)y'' - xy' + y = 0$ becomes

$$x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 + a_0 + 6a_3x - a_1x + a_1x + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2}$$

$$- \sum_{n=2}^{\infty} na_n x^n + \sum_{n=2}^{\infty} a_n x^n$$

$$= 2a_2 + a_0 + 6a_3x + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$- \sum_{n=2}^{\infty} na_n x^n + \sum_{n=2}^{\infty} a_n x^n$$

$$= 2a_2 + a_0 + 6a_3x1 + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$- \sum_{n=2}^{\infty} na_n x^n + \sum_{n=2}^{\infty} a_n x^n$$

$$= 2a_2 + a_0 + 6a_3x$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n(n-1) - (n-1)a_n\right] x^n$$

$$= 2a_2 + a_0 + 6a_3x$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n-1)^2 a_n\right] x^n = 0.$$

So $a_2 = -\frac{1}{2}a_0$, $a_3 = 0$ and

$$a_{n+2} = -\frac{(n-1)^2}{(n+1)(n+2)} a_n, \; n \geq 2.$$

Clearly $a_5 = \frac{(3-1)^2}{4.5} a_3 = 0$ and in fact all odd coefficients higher than one are equal to zero. So $y = a_1 x$ is a solution. This is obvious. We can generate the other coefficients as follows.

$$a_4 = -\frac{(2-1)^2}{3.4} a_2 = (-1)^2 \frac{1}{2 \times 3 \times 4} a_0,$$

$$a_6 = -\frac{(4-1)^2}{5.6} a_4 = (-1)^3 \frac{3^2}{6!} a_0$$

In general the even coefficients are given by

$$a_{2n} = (-1)^n \frac{1 \times 1 \times 3^2 \times 5^2 \times \cdots (2n-3)^2}{(2n)!} a_0, n \geq 1.$$

So the second solution is

$$y = a_0(1 + \sum_{n=0}^{\infty} (-1)^n \frac{1 \times 1 \times 3^2 \times 5^2 \times \cdots (2n-3)^2}{(2n)!} x^{2n}.$$

Now we have an explicit solution $y = x$. Using the methods from lectures we can construct a second solution. It is

$$y = \sinh^{-1} x - \sqrt{x^2+1}.$$

This is the function the power series represents.

(i) Here we have the most challenging problem. $y'' + e^x y' + (x^2+1)y = 0$ and we want $y(0) = 1$, $y'(0) = 0$. Now if $y = \sum_{n=0}^{\infty} a_n x^n$ then $y(0) = a_0 = 1$. Also $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$. So $y'(0) = a_1 = 0$. So we know that $a_1 = 0$. Substituting the series into the equation gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=1}^{\infty} n a_n x^{n-1} + (x^2+1) \sum_{n=0}^{\infty} a_n x^n = 0.$$

We proceed as with problem (e). I will not go through the details. If you want them, I will provide them on a separate sheet, since the details are messy. This leads to a series of equations for the coefficients. These are

$$\begin{aligned} 2a_2 + a_1 + a_0 &= 0 \\ 6a_3 + 2a_1 + 2a_2 &= 0 \\ 12a_4 + 3a_3 + 3a_2 + \frac{1}{2}a_1 + a_0 &= 0. \end{aligned}$$

etc. Solving these gives the solution

$$y = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{1}120x^5 + \frac{11}{720}x^6 + \cdots .$$