37335 Differential Equations.

Tutorial Six Solutions.

Q1. Solve Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

We set

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Proceeding as usual we get

$$(1-x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^k =$$

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} n(n+1)a_k x^k$$

$$= 2a_2 + n(n+1)a_0 + (6a_3 - a_1(2 - n(n+1)))x + \sum_{k=4}^{\infty} k(k-1)a_k x^k$$

$$-\sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=2}^{\infty} 2k a_k x^k + \sum_{k=2} n(n+1)a_k x^k$$

$$= 2a_2 + n(n+1)a_0 + (6a_3 - a_1(2 - n(n+1)))x$$

$$+\sum_{k=2}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty}(k^2+k-n^2-n)a_k x^k$$

$$= 2a_2 + n(n+1)a_0 + (6a_3 - a_1(2 - n(n+1)))x$$

$$+\sum_{k=2}^{\infty}\left[(k+2)(k+1)a_{k+2} - (k^2+k-n^2-n)a_k\right]x^k = 0$$

So $a_2 = -\frac{n(n+1)}{2}a_0$, $a_3 = \frac{2-n(n+1)}{6}a_1$ and

$$a_{k+2} = \frac{k^2+k-n^2-n}{(k+2)(k+1)}a_k.$$

Assume that n is a positive integer. Obviously $a_{n+2} = \frac{n^2+n-n^2-n}{(n+2)(n+1)}a_n = 0$. It then follows that $a_{n+4} = 0, a_{n+6} = 0$ etc. So if n is an integer the coefficient a_{n+2j} will be zero for all $j = 1, 2, \ldots$. Suppose that $n = 4$. Then one of the solutions will be

$$y = a_0 + a_2x^2 + a_4x^2,$$

which is a polynomial of degree 4. All the other even terms will be zero. If $n = 5$ the solution will be

$$y = a_1x + a_3x^2 + a_5x^5,$$

which is again a polynomial. The coefficients a_7, a_9 etc will all be zero, so the solution terminates after the fifth term of the series. The polynomials are called Legendre polynomials.

Finding closed form expressions for the coefficients a_k in the general solution requires a bit of work. One formula for the polynomial solutions is

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k.$$

For n not an integer, the solutions are given by what are called Legendre functions. You can generate terms in the series expansions from the recurrence formulae, but these are not terribly useful. There are more useful expressions. For example if $n = s$, s real, we have

$$P_s(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x + \sqrt{x^2-1}\cos\theta)^s d\theta. \tag{0.1}$$

There is a similar formula for the second solution. You can find it in various books. We will not make any further use of them, but Legendre polynomials are used extensively in, for example, numerical integration. Schemes which are more accurate than Simpson's rule can be obtained using Legendre polynomials. In fact such highly accurate schemes can obtained using other families of polynomials, but this is outside the scope of our subject.

Question 2.

Here we solve Gauss' hypergeometric equation

$$x(1-x)y'' + [c - (1+a+b)x]y' - aby = 0.$$

We look for a solution of the form $y = \sum_{n=0}^{\infty} A_n x^{n+s}$. Substituting into the equation gives

$$x(1-x)\sum_{n=0}^{\infty} A_n(n+s)(n+s-1)x^{n+s-2}$$
$$+ (c-(1+a+b)x)\sum_{n=0}^{\infty} A_n(n+s)x^{n+s-1} - ab\sum_{n=0}^{\infty} A_n x^{n+s} = 0.$$

Which is

$$\sum_{n=0}^{\infty} A_n(n+s)(n+s-1)x^{n+s-1} - \sum_{n=0}^{\infty} A_n(n+s)(n+s-1)x^{n+s}$$
$$+ \sum_{n=0}^{\infty} cA_n(n+s)x^{n+s-1} - \sum_{n=0}^{\infty}(1+a+b)(n+s)A_n x^{n+s} - \sum_{n=0}^{\infty} abA_n x^{n+s}$$
$$= 0.$$

Or

$$(s(s-1)+sc)A_0x^{s-1}+\sum_{n=1}^{\infty}A_n(n+s)(n+s-1)x^{n+s-1}$$
$$-\sum_{n=0}^{\infty}A_n(n+s)(n+s-1)x^{n+s}$$
$$+\sum_{n=1}^{\infty}cA_n(n+s)x^{n+s-1}-\sum_{n=0}^{\infty}(1+a+b)(n+s)A_nx^{n+s}-\sum_{n=0}^{\infty}abA_nx^{n+s}$$
$$=0.$$

This gives $s=0$ or $s=1-c$ and

$$\sum_{n=1}^{\infty}A_n(n+s)(n+s-1)x^{n+s-1}-\sum_{n=1}^{\infty}A_{n-1}(n+s-1)(n+s-2)x^{n+s-1}$$
$$+\sum_{n=1}^{\infty}cA_n(n+s)x^{n+s-1}-\sum_{n=1}^{\infty}(1+a+b)(n+s-1)A_{n-1}x^{n+s-1}$$
$$-\sum_{n=1}^{\infty}abA_{n-1}x^{n+s-2}$$
$$=0.$$

Hence

$$\sum_{n=1}^{\infty}[(n+s)(n+s-1+c)A_n-((n+s-1)(n+s-2)$$
$$+((n+s-1)(1+a+b)+ab)A_{n-1}]x^{n+s-1}=0$$

Hence

$$A_n=\frac{(n+s-1)((n+s-2)+(1+a+b))+ab)}{(n+s)(n+s-1+c)}A_{n-1},\ n\geq 1.$$

Take $s=0$. Then

$$\begin{aligned}A_n&=\frac{(n-1)[(n-2)+(1+a+b)]+ab)}{n(n-1+c)}A_{n-1}\\&=\frac{(n-1)[(n-1+a+b)]+ab}{n(n-1+c)}A_{n-1}\\&=\frac{(n+a-1)(n+b-1)}{n(n-1+c)}A_{n-1},\ n\geq 1.\end{aligned}$$

Thus $A_1=\frac{ab}{c}A_0$.

$$A_2=\frac{1+a+b+ab}{2(c+1)}A_1=\frac{(1+a)(1+b)}{2(c+1)}A_1=\frac{a(a+1)b(b+1)}{2c(c+1)}A_0$$

$$A_3 = \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}$$

etc.

Introduce the rising factorial or Pochhammer symbol $(a)_n = a(a+1).\cdots(a+n-1)$. Then the solution for $s=0$ can be written in terms of Gauss' Hypergeometric function

$$_2F_1(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n.$$

This is valid if c is not a negative integer. For c a negative integer we can define the function using an integral representation, but we will not discuss this. For $s = 1-c$ a similar calculation gives the second solution

$$y_2 = x^{1-c} {}_2F_1(1+a-c, 1+b-c, 2-c, x)$$

Hypergeometric functions play a major role in the study of differential equations. Many ODEs can be converted into a hypergeometric equation by a change of variables and many different functions are special cases of hypergeometric functions.

Question 3. We have the so called confluent hypergeometric equation $xy''+(b-x)y'-ay=0$. As usual put $y=\sum_{n=0}^{\infty} A_n x^{n+s}$ and substitute into the equation to obtain

$$\begin{aligned}
x\sum_{n=0}^{\infty}&(n+s)(n+s-1)A_n x^{n+s-2} + (b-x)\sum_{n=0}^{\infty}(n+s)A_n x^{n+s-1} \\
&- a\sum_{n=0}^{\infty} A_n x^{n+s} = \sum_{n=0}^{\infty}(n+s)(n+s-1)A_n x^{n+s-1} \\
&+ \sum_{n=0}^{\infty} b(n+s)A_n x^{n+s-1} - \sum_{n=0}^{\infty}(n+s)A_n x^{n+s} - \sum_{n=0}^{\infty} aA_n x^{n+s} \\
=& (s(s-1)+bs)A_0 x^{s-1} + \sum_{n=1}^{\infty}[(n+s)(n+s-1)+b(n+s)]A_n x^{n+s-1} \\
&- \sum_{n=0}^{\infty}(n+s+a)A_n x^{n+s} \\
=& (s^2+(b-1)s)A_0 x^{s-1} \\
&\quad + \sum_{n=1}^{\infty}[(n+s)(n+s-1+b)A_n - (n+s+a-1)A_{n-1}]x^{n+s-1} = 0.
\end{aligned}$$

So $s=0$ or $1-b$ and

$$A_n = \frac{(n+s+a-1)}{(n+s)(n+s-1+b)} A_{n-1}, \ n \geq 1.$$

We introduce the rising factorial, which is also called Pochhammer's symbol. $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$. Then

$A_1 = \frac{a}{1\times b}A_0, A_2 = \frac{a+1}{1\times 2b}A_1 = \frac{a(a+1)}{2!(b(b+1)}A_0, A_3 = \frac{a(a+1)(a+2)}{3!b(b+1)(b+2)}$ etc. In general we have

$$A_n = \frac{(a)_n}{n!(b)_n}$$

and the solution is

$$y = A_0 \sum_{n=0}^{\infty} \frac{(a)_n}{n!(b)_n} x^n.$$

The choice $A_0 = 1$ produces Kummer's confluent hypergeometric function, which is usually written as

$${}_1F_1(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(b)_n} x^n.$$

There is a second solution which is called Tricomi's confluent hypergeometric function. This will be derived in the workshop. It can be written as the Laplace transform

$$U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xt} t^{a-1} (1+t)^{a+b-1} dt.$$

A wide range of functions are actually special cases of the hypergeometric functions.

Question 4. The Hermite equation is $y'' - 2xy' + 2ny = 0$. We have only ordinary points, so put $y = \sum_{k=0}^{\infty} a_k x^k$. So

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=1}^{\infty} 2ka_k x^k + \sum_{k=0}^{\infty} 2na_k x^k$$

$$= 2a_2 + 2na_0 + \sum_{k=3}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=1}^{\infty} 2ka_k x^k + \sum_{k=1}^{\infty} 2na_k x^k$$

$$= 2a_2 + 2na_0 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=1}^{\infty} 2ka_k x^k + \sum_{k=1}^{\infty} 2na_k x^k$$

$$= 2a_2 + 2na_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} - 2(k-n)a_k] x^k$$

Thus $a_2 = -na_0$ and

$$a_{k+2} = \frac{2(k-n)}{(k+2)(k+1)} a_k, \ k \geq 1.$$

Clearly if n is a positive integer then

$$a_{n+2} = \frac{2(n-n)}{(n+2)(n+1)} a_n = 0.$$

So all further terms a_{n+4}, a_{n+6} etc will be zero. Thus one of the solutions will be a polynomial of degree n. These are known as Hermite polynomials. We can generate this for different values of n. If $n=0$, $y=1$, if $n=1$ we get $y=2x$ is a solution. For $n=2$ we get $y=4x^2-2$. We can prove that the nth Hermite polynomial written $H_n(x)$ is given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

However we will not give a rigourous proof, but merely point out that if you test for various values of n, you can check that it produces polynomial solutions of Hermite's equation. Finally if we expand as a Taylor series we get

$$\begin{aligned} f(x,t) = e^{2xt-t^2} &= f(x,0)t + f_t(x,0)t + \frac{1}{2}f_{tt}(x,0)t^2 + \frac{1}{3!}f_{ttt}(x,0)t^3 + \cdots \\ &= 1 + 2xt + \frac{1}{2}(4x^2-2)t^2 + \frac{1}{3!}(8x^3-12x)t^3 + \cdots \\ &= \sum_{n=}^{\infty} \frac{1}{n!}H_n(x)t^n. \end{aligned}$$

Questions 5 and 6. The solutions are in the lecture notes in the chapter on Bessel functions.

Question 7.

We have a differential equation $u'' + a_1(x)u' + a_2(x)u = 0$. A common trick is to knock out the middle term. There are several ways of doing this, but they are all basically the same idea, just presented differently. Let us try the substitution $u = e^{\int \phi} v$. Put $\psi = \int \phi$. Then $u = e^{\psi} v$.

Now by the product rule

$$u' = \psi' e^{\psi} v + e^{\psi} v' = e^{\psi}(\psi' v + v')$$

and

$$u'' = e^{\psi}(v'' + 2\psi' v' + (\psi'' + (\psi')^2)v).$$

$$\begin{aligned} u'' + a_1(x)u' + a_2(x)u &= e^{\psi}(v'' + 2\psi' v' + (\psi'' + (\psi')^2)v) \\ &\quad + a_1(x)e^{\psi}(\psi' v + v') + a_2(x)e^{\psi} v \\ &= e^{\psi}(v'' + (2\psi' + a_1(x))v' + (\psi'' + (\psi')^2 + a_1(x)\psi' + a_2(x))v = 0. \end{aligned}$$

Thus we have to solve

$$v'' + (2\psi' + a_1(x))v' + (\psi'' + (\psi')^2 + a_1(x)\psi' + a_2(x))v = 0.$$

If we put $2\psi' = -a_1(x)$ we get

$$v'' + (\psi'' + (\psi')^2 + a_1(x)\psi' + a_2(x))v = 0$$

and letting $\psi = \int \phi$ gives the result. We could also knock out the lowest order term by setting $\psi'' + (\psi')^2 + a_1(x)\psi' + a_2(x) = 0$, but it is much harder to find ψ in this case. Actually finding a ψ which knocks out the lowest derivative is equivalent to solving the original equation.

Question 8.
a) We want to solve $u'' + 2\cot x u' - u = 0$. We let $\psi' = -\cot x$. Then $\psi'' = \mathrm{cosec}^2 x$. The equation we have to solve is then

$$v'' + (\mathrm{cosec}^2 x + \cot^2 x - 2\cot^2 x - 1)v = v'' = 0.$$

So $v(x) = Ax + B$. Now $\psi' = -\cot x$, hence $\psi = -\ln(\sin x)$. Thus the solution of the original problem is

$$u(x) = e^{-\ln(\sin x)}(Ax + B) = \frac{Ax + B}{\sin x}.$$

b) The equation to solve is $u'' + 2\tan x u' + 2\tan^2 x u = 0$. We put $\psi' = -\tan x$. Then $\psi'' = -\sec^2 x$. We are then led to

$$v'' + (-\sec^2 x + \tan^2 x - 2\tan^2 x + 2\tan^2 x)v = v'' - v = 0.$$

So $v(x) = Ae^x + Be^{-x}$. Since $\psi' = -\tan x$, $\psi = \ln(\cos x)$. So the solution of the original equation is

$$u(x) = \cos x(Ae^x + Be^{-x}).$$

This is surprisingly useful trick. It has been used to reduce ODEs to more tractable problems for a couple of centuries.