37335 Differential Equations.

Tutorial Three Solutions.

Question 1.

Constructing a Second Solution From a Known Solution.

(a) We have the equation $x^3y'' + xy' - y = 0$. We know that $y_1(x) = x$ is a solution, since $y_1' = 1, y_1'' = 0$, so $x^3y_1'' + xy_1' - y_1 = 0 + x - x = 0$.

We put the equation into the standard form so that we can use our formula. We have $y'' + \frac{1}{x^2}y' - \frac{1}{x^3}y = 0$. It is clear that $p(x) = \frac{1}{x^2}$. We easily see that $\int p(x)dx = -\frac{1}{x}$. Now the second solution $y_2$ is given by

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2}dx = x \int \frac{e^{\frac{1}{x}}}{x^2}dx.$$

To evaluate this make the change of variables $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}dx$ so that the integral becomes

$$\int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int e^u du = -e^u = -e^{\frac{1}{x}}.$$

Since any multiple of a solution is a solution, we can drop the minus sign. Then a second solution is $y_2 = xe^{\frac{1}{x}}$.

(b) $xy'' + (1 - 2x)y' + (x - 1)y = 0$ has a solution $y_1 = e^x$. Clearly $p(x) = \frac{1-2x}{x} = \frac{1}{x} - 2$. Integrating we have $\int p(x)dx = \int \left(\frac{1}{x} - 2\right) dx = \ln x - 2x$. The second solution is then

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2}dx = e^x \int \frac{e^{-\ln x + 2x}}{e^{2x}}dx \\ &= e^x \int \frac{1}{x}dx = e^x \ln x. \end{aligned}$$

(c) The equation $2xy'' + (1-4x)y' + (2x-1)y = e^x$, is inhomogeneous. The homogeneous problem has a solution $y_1 = e^x$. We first construct a second solution. Now $p(x) = \frac{1-4x}{2x} = \frac{1}{2x} - 2$. Then $\int p(x)dx = \int (\frac{1}{2x} - 2)dx = \frac{1}{2}\ln x - 2x$. A second solution is then

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2}dx = e^x \int \frac{e^{-\frac{1}{2}\ln x + 2x}}{e^{2x}}dx \\ &= e^x \int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}e^x. \end{aligned}$$

Let $y_2 = 2\sqrt{x}e^x$. We see that $R(x) = \dfrac{1}{2x}e^x$. The Wronskian is

$$\begin{aligned} W(y_1, y_2) = y_1 y_2' - y_2 y_1' &= e^x(\frac{1}{\sqrt{x}}e^x + \sqrt{x}e^x) - \sqrt{x}e^x e^x \\ &= \frac{e^{2x}}{\sqrt{x}}. \end{aligned}$$

A particular solution is given by $y_p = uy_1 + vy_2$ in which

$$u' = -\frac{y_2 R}{W}, \ v' = \frac{y_1 R}{W}.$$

So

$$u = -\int \left(2\sqrt{x}e^x \frac{1}{2x}e^x\right) / \left(\frac{e^{2x}}{\sqrt{x}}\right) dx = -\int dx = -x$$

and

$$v = \int \left(e^x \frac{1}{2x}e^x\right) / \left(\frac{e^{2x}}{\sqrt{x}}\right) dx = \int \frac{1}{2\sqrt{x}}dx = \sqrt{x}.$$

Thus $y_p = -xe^x + \sqrt{x}(2\sqrt{x}e^x) = xe^x$.

(d) So we have $x^2(x+2)y'' + 2xy' - 2y = (x+2)^2$. $y_1 = x$ is a solution of the associated homogeneous equation. We find $p(x) = \frac{2}{x(x+2)} = \frac{1}{x} - \frac{1}{x+2}$. So $\int p(x)dx = \ln x - \ln(x+2) = \ln\left(\frac{x}{x+2}\right)$. Hence $e^{-\int p(x)dx} = \frac{x+2}{x} = 1 + \frac{2}{x}$. So a second solution of the inhomogeneous problem is given by

$$\begin{aligned} y_2 = x\int (1+\frac{2}{x})\frac{1}{x^2}dx &= x\int \left(\frac{1}{x^2} + \frac{2}{x^3}\right) dx \\ &= -x\left(\frac{1}{x} + \frac{1}{x^2}\right) = -1 - \frac{1}{x}. \end{aligned}$$

Let $y_2 = 1 + \frac{1}{x}$. Now $R(x) = \dfrac{x+2}{x^2}$. The Wronskian is

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = -(1 + \frac{2}{x}).$$

We then have

$$\begin{aligned} u' = -\frac{y_2 R}{W} &= (1+\frac{1}{x})\frac{x+2}{x^2}/(1+\frac{2}{x}) \\ &= \frac{x+1}{x^2}. \end{aligned}$$

Hence $u = \ln x - \frac{1}{x}$. For $v$ we find

$$v' = \frac{y_1 R}{W} = -x(\frac{x+2}{x^2})/(1+\frac{2}{x}) = -1.$$

So $v = -x$. Then the particular solution is

$$y_p = uy_1 + vy_2 = x\ln x - x - 2.$$

These two examples show that in theory, if we have a single solution of the equation $y''+a(x)y'+b(x)y=0$, then we can construct a solution of the inhomogeneous problem $y''+a(x)y'+b(x)y=R(x)$.

Question 2.

Variation of Parameters Continued.

(a) We solve $y''-3y'+2y=\frac{-e^{2x}}{e^x+1}$. We take $y_1=e^x, y_2=e^{2x}$. The Wronskian is

$$W(y_1,y_2)=y_1y_2'-y_2y_1'=e^{3x}.$$

Clearly $R(x)=\frac{-e^{2x}}{e^x+1}$. So

$$u'=-\frac{y_2R}{W}=\frac{e^{4x}}{e^{3x}(e^x+1)}=\frac{e^x}{e^x+1}.$$

Integrating gives $u=\int\frac{e^x}{e^x+1}dx=\int\frac{dw}{w+1}=\ln(w+1)=\ln(e^x+1)$ where we used the obvious substitution $w=e^x$. Also

$$\begin{aligned} v'=\frac{y_1R}{W} &= -\frac{e^{3x}}{e^{3x}(e^x+1)} \\ &= -\frac{1}{e^x+1} \\ &= -\frac{e^{-x}}{e^{-x}+1}. \end{aligned}$$

So $v=\ln(e^{-x}+1)$. Thus $y_p=e^x\ln(e^x+1)+e^{2x}\ln(e^{-x}+1)$.

(b) $y''+y=\tan x\sec x$. Take $y_1=\cos x$ and $y_2=\sin x$. Then $W(y_1,y_2)=\cos^2x+\sin^2x=1$. Then

$$u'=-\frac{y_2R}{W}=-\sin x\tan x\sec x=-\tan^2x=1-\sec^2x.$$

Hence $u=x-\tan x$.

$$v'=\frac{y_1R}{W}=\cos x\tan x\sec x=\tan x.$$

So $v=\int\frac{\sin x}{\cos x}dx=-\ln(\cos x)=\ln(\sec x)$. Giving

$$y_p=\cos x(x-\tan x)+\sin x\ln(\sec x)=x\cos x-\sin x+\sin x\ln(\sec x).$$

(c) $y''+2y'+y=e^{-x}\sec^2x$. $y''+2y'+y=0$ has solutions $y_1=e^{-x}$ and $y_2=xe^{-x}$. $R(x)=e^{-x}\sec^2x$. Now

$$W(y_1,y_2)=y_1y_2'-y_1'y_2=e^{-x}(e^{-x}-xe^{-x})+xe^{-x}e^{-x}=e^{-2x}.$$

Then

$$u'=-\frac{y_2R}{W}=-xe^{-x}e^{-x}\sec^x/(e^{-2x})=-x\sec^2x.$$

So that

$$u = -\int x\sec^2 x dx = -x\tan x + \int \tan x dx$$
$$= -x\tan x + \ln(\sec x).$$

$$v' = \frac{y_1 R}{W} = e^{-x}e^{-x}\sec^2 x/(e^{-2x}) = \sec^2 x.$$

Hence $v = \tan x$. Hence

$$y_p = e^{-x}(\ln(\sec x) - x\tan x) + xe^{-x}\tan x = e^{-x}\ln(\sec x).$$

(d) $y'' - y = \frac{2}{e^x+1}$. Take $y_1 = e^x, y_2 = e^{-x}$. Obviously $R(x) = \frac{2}{e^x+1}$ and

$$W(y_1, y_2) = e^x(-e^x) - e^x(e^x) = -2.$$

Thus

$$u' = -\frac{2e^{-x}}{-2(e^x+1)} = \frac{e^{-x}}{e^x+1} = \frac{1}{e^x(e^x+1)}$$
$$= \frac{1}{e^x} - \frac{1}{e^x+1} = e^{-x} - \frac{e^{-x}}{e^{-x}+1}.$$

Integrating gives $u = -e^{-x} + \ln(e^{-x}+1)$. Next

$$v' = \frac{y_1 R}{W} = \frac{2e^x}{-2(e^x+1)} = \frac{-e^x}{e^x+1}.$$

Hence $v = -\ln(e^x+1)$. Hence

$$y_p = e^x(-e^{-x} + \ln(e^{-x}+1)) + e^{-x}(-\ln(e^x+1))$$
$$= e^x\ln(e^{-x}+1) - 1 - e^{-x}\ln(e^x+1).$$

(e) $y'' + 2y' + y = 4e^{-x}\ln x$. We have seen the homogeneous problem before. Take $y_1 = e^x$, $y_2 = xe^{-x}$. Also $W(y_1, y_2) = e^{-2x}$. Now

$$u' = -\frac{y_2 R}{W} = -\frac{4xe^{-2x}\ln x}{e^{-2x}} = -4x\ln x.$$

We integrate by parts to obtain

$$u = -4\int x\ln x dx = -2x^2\ln x + 2\int x dx = -2x^2\ln x + x^2.$$

Next we find $v$.

$$v' = \frac{y_1 R}{W} = 4e^{-2x}\ln x/(e^{-2x}) = 4\ln x.$$

Integrating we have $v = \int 4\ln x dx = 4x\ln x - \int 4dx = 4x\ln x - 4x$. So we have

$$y_p = e^{-x}(-2x^2\ln x + x^2) + xe^{-x}(4x\ln x - 4x)$$
$$= 2x^2e^{-x}\ln x - 3x^2e^{-x}.$$

(f) $y'' + y = \text{cosec} x$. With $y_1 = \cos x, y_2 = \sin x$ and $W(y_1, y_2) = 1$ we have

$$u' = -\sin x \text{cosec} x = -1,$$

giving $u = -x$.

$$v' = \cos x \text{cosec} x = \cot x.$$

So $v = \ln(\sin x)$. Hence $y_p = -x\cos x + \sin x \ln(\sin x)$.

(g) $y'' - 2ay' + (a^2 + b^2)y = e^{ax}(A\cos(bx) + B\sin(bx)) = R(x)$. The solutions for the homogeneous problem are $y_1 = e^{ax}\cos(bx), y_2 = e^{ax}\sin(bx)$. Then

$$\begin{aligned} W(y_1, y_2) &= e^{ax}\cos(bx)(ae^{ax}\sin(bx) + be^{ax}\cos(bx) \\ &\quad - e^{ax}\sin(bx)(ae^{ax}\cos(bx) - be^{x}\sin(bx) \\ &= be^{2ax}. \end{aligned}$$

After some algebra we find

$$u' = -\frac{1}{b}(B\sin^2(bx) + A\sin(bx)\cos(bx))$$

Now $\sin(bx)\cos(bx) = \frac{1}{2}\sin(2ax)$, $\sin^2(bx) = \frac{1}{2}(1 - \cos(2bx))$. Hence

$$u' = -\frac{B}{2b}(1 - \cos(2bx)) - \frac{A}{2b}\sin(2bx),$$

or

$$u = -\frac{Bx}{2b} + \frac{B}{4b^2}\sin(2bx) + \frac{A}{4b^2}\cos(2bx).$$

For $v$

$$\begin{aligned} v' &= \frac{1}{b}(A\cos^2 bx) + B\sin(bx)\cos(bx)) \\ &= \frac{A}{2b}(1 + \cos(2bx)) + \frac{B}{2b}\sin(2bx). \end{aligned}$$

Hence

$$v = \frac{Ax}{2b} + \frac{A}{4b^2}\sin(2bx) - \frac{B}{4b^2}\cos(2bx).$$

So we have

$$\begin{aligned} y_p &= e^{ax}\cos(bx)\left(-\frac{Bx}{2b} + \frac{B}{4b^2}\sin(2bx) + \frac{A}{4b^2}\cos(2bx)\right) \\ &\quad + e^{ax}\sin(bx)\left(\frac{Ax}{2b} + \frac{A}{4b^2}\sin(2bx) - \frac{B}{4b^2}\cos(2bx)\right). \end{aligned}$$

Question 3

General Solution Formulae

We have the equation $y'' + (a+b)y' + aby = F(x)$. We suppose that $a \neq b$. The homogeneous problem has solutions $y_1 = e^{-ax}, y_2 = e^{-bx}$. The Wronskian is then

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' = -be^{-ax}e^{-bx} - (-ae^{-ax}e^{-bx}) \\ &= (a-b)e^{-(a+b)x}. \end{aligned}$$

Now

$$u' = -\frac{y_2 F}{W} = -\frac{e^{-bx}F(x)}{(a-b)e^{-(a+b)x}} = -\frac{1}{(a-b)}e^{ax}F(x).$$

Recall that if $G$ is the antiderivative of $g$, then $w(x) = \int g(x)dx = G(x) + C$ where $C$ is a constant on integration. This is equivalent to writing $w(x) = \int_{x_0}^{x} g(t)dt$ for some starting point $x_0$. This is just the fundamental Theorem of Calculus, since $\int_{x_0}^{x} g(t) = [G(t)]_{x_0}^{x} = G(x) - G(x_0)$. So the constant of integration $C$ is in fact equal to $-G(x_0)$.

Using this fact we can write $u$ as a *definite* integral. This gives

$$u(x) = -\int_{x_0}^{x} \frac{1}{(a-b)} e^{at} F(t)dt.$$

Here $x_0$ is an arbitrary starting point. We do the same for $v$.

$$\begin{aligned} v' &= \frac{y_1 F}{W} = \frac{e^{-ax}F(x)}{(a-b)e^{-(a+b)x}} \\ &= \frac{e^{bx}}{a-b}F(x). \end{aligned}$$

As with $u$ we write

$$v = \frac{1}{a-b}\int_{x_0}^{x} e^{bt}F(t)dt.$$

Hence

$$\begin{aligned} y_p &= e^{ax}u(x) + e^{bx}v(x) \\ &= -e^{-ax}\int_{x_0}^{x} \frac{e^{at}}{(a-b)}F(t) + e^{bx}\int_{x_0}^{x} \frac{e^{bt}}{a-b}F(t)dt \\ &= \int_{x_0}^{x} \left[\frac{e^{-b(x-t)} - e^{-a(x-t)}}{a-b}\right] F(t)dt. \end{aligned}$$

Question 4.

We can construct a general formula as follows. We have the ODE $y'' + p(x)y' + q(x)y = F(x)$. Two linearly independent solutions of the inhomogeneous problem are $y_1$ and $y_2$. The Wronskian is $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ Then

$$u' = -\frac{y_2 F}{W},$$

so

$$u(x) = -\int_{x_0}^{x} \frac{y_2(t)F(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)} dt.$$

Similarly

$$v' = \frac{y_1 F}{W},$$

giving

$$v(x) = \int_{x_0}^{x} \frac{y_1(t)F(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)} dt.$$

Combining we have the particular solution

$$y_p(x) = \int_{x_0}^{x} \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)} F(t)dt.$$

Question 5.

Variation of Parameters for Higher Order Equations.

(a) The equation is $x^3y''' + x^2y'' - 2xy' + 2y = x^3 \sin x$. The homogeneous equation is of Euler type and its solutions are of the form $y = x^a$. So we substitute this into the homogeneous equation to get $x^a(a(a-1)(a-2) + a(a-1) - 2a + 1) = 0$. It is easy to check that this cubic has roots $\pm 1$ and 2. So there are three linearly independent solutions $y_1 = x, y_2 = 1/x$ and $y_3 = x^2$.

A laborious calculation with a three by three determinant gives $W(y_1, y_2, y_3) = -\frac{6}{x}$. The sub-Wronskians are

$$\begin{aligned} W_1(x) &= (-1)^{3-1}W(y_2, y_3) = 3, \\ W_2(x) &= (-1)^{3-2}W(y_1, y_3) = -x^2 \\ W_3(x) &= (-1)^{3-2}W(y_1, y_2) = -\frac{2}{x}. \end{aligned}$$

Now $R(x) = \sin x$ (Divide equation by $x^3$) From the given formulae we have

$$v_1'(x) = \frac{W_1 R}{W(y_1, y_2, y_3)} = -\frac{1}{2} x \sin x.$$

Integration gives $v_1(x) = \frac{1}{2}(x \cos x - \sin x)$. Similarly

$$v_2'(x) = \frac{W_2(x)R(x)}{W(y_1, y_2, y_3)} = \frac{1}{6} x^3 \sin x.$$

Hence

$$v_2(x) = \frac{1}{6}(3(x^2 - 2) \sin x - x(x^2 - 6) \cos x).$$

Similarly we have $v_3'(x) = \frac{1}{3}\sin x$ giving $v_3(x) = -\frac{1}{3}\cos x$. Finally we have

$$\begin{aligned} y_p(x) &= y_1(x)v_1(x) + y_2(x)v_2(x) + y_3(x)v_3(x) \\ &= \frac{1}{2}x(x\cos x - \sin x) + \frac{1}{6x}(3(x^2-2)\sin x - x(x^2-6)\cos x) \\ &\quad - \frac{x^2}{3}\cos x \\ &= \cos x - \frac{1}{x}\sin x. \end{aligned}$$

(b) We have the fourth order equation $y^{(iv)} - k^2 y'' = g(x), 0 < x < L$. First we solve $y^{(iv)} - k^2 y'' = 0$. This leads to four solutions $y_1 = 1, y_2 = x, y_3 = e^{kx}, y_4 = e^{-kx}$. It is easy to check that these are correct. We then have to compute the Wronskian and all the various sub-Wronskians. The Wronskian is given by a four by four determinant. The calculations are tedious and we will only present the results. It is best to do them in Mathematica. However we find

$$W(y_1, y_2, y_3, y_4) = 2k^5.$$

Then

$$\begin{aligned} W_1(x) &= (-1)^{4-1}W(y_2, y_3, y_4) = 2k^3 x, \\ W_2(x) &= (-1)^{4-2}W(y_1, y_3, y_4) = -2k^3 \\ W_3(x) &= (-1)^{4-3}W(y_1, y_2, y_4) = -k^2 e^{kx} \\ W_4(x) &= (-1)^{4-4}W(y_1, y_2, y_3) = k^2 e^{-kx}. \end{aligned}$$

Then

$$\begin{aligned} v_1'(x) &= \frac{W_1(x)g(x)}{W(y_1, y_2, y_3, y_4)} = \frac{xg(x)}{k^2}, \\ v_2'(x) &= \frac{W_2(x)g(x)}{W(y_1, y_2, y_3, y_4)} = \frac{-g(x)}{k^2}, \\ v_3'(x) &= \frac{W_3(x)g(x)}{W(y_1, y_2, y_3, y_4)} = \frac{-e^{-kx}g(x)}{2k^3}, \\ v_4'(x) &= \frac{W_4(x)g(x)}{W(y_1, y_2, y_3, y_4)} = \frac{e^{kx}g(x)}{2k^3}. \end{aligned}$$

Then the general solution is

$$\begin{aligned} y &= \sum_{n=1}^{4} c_n y_n(x) + \frac{1}{k^3}\int_0^x \left( kt - kx - \frac{1}{2}e^{kx}e^{-kt} + \frac{1}{2}e^{-kx}e^{kt} \right) g(t)dt \\ &= \sum_{n=1}^{4} c_n y_n(x) + \int_0^x \left( \frac{(t-x)}{k^2} - \frac{\sinh(k(t-x))}{k^3} \right) g(t)dt. \end{aligned}$$

Question 6. This is an assignment question. The solution will become available after the assignment is submitted.