37335 Differential Equations.

Tutorial Two Solutions.

Linear Constant Coefficient Equations.

Question 1.
(a) We solve $2y'' + 4y' + 8y = 0$. This is the same as $y'' + 2y' + 4y = 0$. The auxiliary equation is

$$\lambda^2 + 2\lambda + 4 = 0.$$

The roots are

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \times 4}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = -1 \pm \sqrt{3}i.$$

So we have $y = e^{-x}\left(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right)$.

(b) The equation $y'' + 16y = x$ is inhomogeneous. First we solve the equation $y'' + 16y = 0$. The auxiliary equation is $\lambda^2 + 16 = 0$. The roots are $\pm 4i$. So the homogenous problem has solution $y_h = c_1 \cos(4x) + c_2 \sin(4x)$. Now we try $y_p = Ax + B$ for a particular solution. Notice $y_p' = A$ and $y_p'' = 0$. Hence we must have $16Ax + 16B = x$. So $B = 0$ and $16A = 1$. Thus the general solution is

$$y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{1}{16}x.$$

(c) Another inhomogenous equation is $y'' - 3y' + 2y = 6e^{-x}$. The auxiliary equation for the homogeneous problem is

$$\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0.$$

Hence the homogenous problem has solution $y_h = c_1 e^{2x} + c_2 e^{x}$. We now look for a particular solution of the form $y_p = Ae^{-x}$. Substituting into the equation we have $Ae^{-x} + 3Ae^{-x} + 2Ae^{-x} = 6Ae^{-x} = 6e^{-x}$. Then $A = 1$ and the general solution is

$$y = c_1 e^{2x} + c_2 e^{x} + e^{-x}.$$

(d) For the equation $y'' + 2y' + 5y = 4e^{-x}\cos(2x)$ the auxiliary equation is $\lambda^2 + 2\lambda + 5 = 0 = (\lambda + 1)^2 + 4 = 0$. So the roots are $-1 \pm 2i$. Hence the general solution of the homogeneous problem is

$$y_h = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)).$$

Now the right hand side is a solution of the homogeneous problem. So we try a solution of the form $y_p = xe^{-x}(A\cos(2x) + B\sin(2x))$. Then $y_p' = e^{-x}(A\cos(2x) + B\sin(2x)) + x\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right)$.

$$\begin{aligned} y_p'' &= 2\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) \\ &\quad + x\frac{d^2}{dx^2}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right). \end{aligned}$$

Consequently

$$
\begin{aligned}
&y_p'' + 2y_p' + 5y_p = 2\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) \\
&+ x\frac{d^2}{dx^2}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) \\
&+ 2\left(e^{-x}(A\cos(2x) + B\sin(2x)) + x\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right)\right) \\
&+ 5xe^{-x}(A\cos(2x) + B\sin(2x)). \\
&= x\Bigg(\frac{d^2}{dx^2}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) + 2\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) + \\
&+ 5(A\cos(2x) + B\sin(2x))\Bigg) + 2\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) \\
&+ 2e^{-x}(A\cos(2x) + B\sin(2x)) \\
&= 2\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) + 2e^{-x}(A\cos(2x) + B\sin(2x)).
\end{aligned}
$$

We used the fact that the terms multiplied by x vanish because $e^{-x}(A\cos(2x)+B\sin(2x))$ is a solution of the homogeneous problem. So when we group the terms with an x in front, they add up to zero.

We then have

$$
\begin{aligned}
&2\frac{d}{dx}\left(e^{-x}(A\cos(2x) + B\sin(2x))\right) + 2e^{-x}(A\cos(2x) + B\sin(2x)) \\
&= e^{-x}(-(2A + B)\sin(2x) - (A - 2B)\cos(2x)) + 2e^{-x}(A\cos(2x) + B\sin(2x)) \\
&= e^{-x}(4B\cos(2x) - 4A\sin(2x)).
\end{aligned}
$$

So $A = 0$ and $B = 1$. Hence $y_p = xe^{-x}\sin(2x)$.

(e) We solve the Euler equation $2x^2y'' - 5xy' + 3y = 0$. This has solutions of the form $y = x^a$. Substitution into the equation gives

$$
\begin{aligned}
2x^2a(a-1)x^{a-2} - 5xax^{a-1} + 3x^a &= x^a(2a^2 - 2a - 5a + 3) \\
&= x^a(2a^2 - 7a + 3) = 0.
\end{aligned}
$$

The roots of $2a^2 - 7a + 3 = 0$ are $\frac{1}{2}$ and 3. So the general solution is $y = c_1x^3 + c_2x^{1/2}$.

Question 2.

Third Order Linear Constant Coefficient Equations.

(a) Third order constant coefficient equations can be solved the same way. We solve $y''' - 6y'' + 11y' - 6y = 0$. We look for solutions of the form $y = e^{\lambda x}$ and this leads to the cubic equation $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$. We can use the cubic formula or solve in Mathematica. However the roots multiply to give 6. Try $\lambda = 1$. Then $1 - 6 + 11 - 6 = 0$. So

$\lambda = 1$ is a solution. Now $6 = 2 \times 3$. So try $\lambda = 2$ and $\lambda = 3$. $2^3 - 6 \times 2^2 + 11 \times 2 - 6 = 0$ and $3^3 - 6 \times 3^2 + 11 \times 3 - 6 = 0$. Both are solutions. General solution of the ODE is then

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

(b) The equation $y''' - 4y' = 0$ is really second order. Put $u = y'$ to convert the equation to $u'' - 4u = 0$. This has the solution $u = c_1 e^{2x} + c_2 e^{-2x}$. Then $y' = u$. So we integrate to obtain $y = \frac{1}{2} c_1 e^x - \frac{1}{2} c_2 e^{-2x} + c_3$.

Question 2.

Ricatti Equations.

(a) The Ricatti equation $f' + \frac{1}{2} f^2 = 2x + 4$. Put $f = a \frac{y'}{y}$. Then $f' = a \frac{y''}{y} - a \left(\frac{y'}{y} \right)^2$. Substituting into the equation produces

$$a \frac{y''}{y} - a \left(\frac{y'}{y} \right)^2 + \frac{1}{2} a^2 \left(\frac{y'}{y} \right)^2 = 2x + 4$$

Put $a = 2$ so that the equation becomes

$$2y'' = (2x + 4)y.$$

This is a form of Airy's equation. We will solve it in the lectures.

(b) For $xf' + 2f^2 + 3xf = 0$. Try a general form $f = A(x) \frac{y'}{y}$ to get $f' = A'(x) \frac{y'}{y} + A(x) \frac{y''}{y} - A(x) \left(\frac{y'}{y} \right)^2$. Hence

$$xf' + 2f^2 + 3xf =$$

$$x \left(A'(x) \frac{y'}{y} + A(x) \frac{y''}{y} - A(x) \left(\frac{y'}{y} \right)^2 \right) + 2A^2(x) \left(\frac{y'}{y} \right)^2 + 3xA(x) \frac{y'}{y}.$$

Put $-xA(x) + 2A^2(x) = 0$, or $A(x) = \frac{1}{2} x$. This leads to the equation $x^2 y'' + (3x^2 + x)y' = 0$. In fact putting $y' = w$ produces a first order linear equation.

(c) $(1 + x^2) f' + 4f^2 = \sin x$. Again set $f = A(x) \frac{y'}{y}$ to get $f' = A'(x) \frac{y'}{y} + A(x) \frac{y''}{y} - A(x) \left(\frac{y'}{y} \right)^2$. Giving

$$(1 + x^2) \left(A'(x) \frac{y'}{y} + A(x) \frac{y''}{y} - A(x) \left(\frac{y'}{y} \right)^2 \right) + 4A^2(x) \left(\frac{y'}{y} \right)^2 = \sin x,$$

which means that we set $4A^2(x) = (1 + x^2) A(x)$, or $A(x) = \frac{1}{4}(1 + x^2)$. This leads

$$\frac{1}{4}(1 + x^2)^2 y'' + \frac{1}{2} x (1 + x^2) y' - (\sin x) y = 0.$$

Question 3.

Exact Equations.

(a) $2x\sin y - y\sin x + (x^2\cos y + \cos x)y' = 0$. This is an exact equation. It can be put in the form

$$(x^2\cos y + \cos x)dy + (2x\sin y - y\sin x)dx = 0.$$

(As an historical aside, this is how Euler wrote first order differential equations.) Now $P_x = 2x\cos y - \sin x$ and $Q_y = 2x\cos y - \sin x$. So the equation is exact. There exists F such that $F_x = (2x\sin y - y\sin x)$ and $F_y = x^2\cos y + \cos x$. Hence

$$F(x,y) = \int(2x\sin y - y\sin x)dx = x^2\sin y + y\cos x + h(y).$$

So that

$$F_y = x^2\cos y + \cos x + h'(y) = x^2\cos y + \cos x.$$

Hence $h' = 0$ and h is a constant. Thus an implicit solution is given by $x^2\sin y + y\cos x + C = 0$

(b)We have $(2xy^3 + 8x)dx + (3x^2y^2 + 5)dy = 0$. Thus $P = 2xy^3 + 8x$, $Q = 3x^2y^2 + 5$ and $P_y = 6xy^2$ and $Q_x = 6xy^2$. Therefore $P_y = Q_x$. So equation is exact. Therefore there is a function F such that $F_x = P$ and $F_y = Q$. So $F(x,y) = \int(2xy^3 + 8x)dx = x^2y^3 + 4x^2 + h(y)$. Then $F_y = 3x^2y^2 + h'(y) = 3x^2y^2 + 5$. Hence $h'(y) = 5$ and so $h(y) = 5y + C$. Thus there is an implicit solution y such that $x^2y^3 + 4x^2 + 5y + C = 0$. Now the solution satisfies $y(2) = -1$. So we let $y = -1, x = 2$ to obtain $4(-1)^3 + 4(2)^2 - 5 + C = 0$. Or $C = -7$. So the solution is given implicitly by $x^2y^3 + 4x^2 + 5y = 7$.

(c) The equation can be written $(x^2e^y + 3e^x)dy + (2xe^y + 3ye^x)dx = 0$. $P = 2xe^y + 3ye^x$ and $Q = x^2e^y + 3e^x$. We see that $P_y = 2xe^y + 3e^x$ and $Q_x = 2xe^y + 3e^x$. So $P_y = Q_x$ and the equation is exact. So there is an F such that $F_x = P$ and $F_y = Q$. Thus we have $F = \int(2xe^y + 3ye^x)dx = x^2e^y + 3ye^x + h(y)$ and $F_y = 2xe^y + 3e^x + h'(y) = Q$. So $h'(y) = 0$ and h is constant. Thus there is an implicit solution y such that $x^2e^y + 3ye^x + C = 0$. Since $y(0) = 1/2$ we have $3(1/2)e^0 + C = 0$. So $C = -3/2$. Solution is given implicitly by $x^2e^y + 3ye^x = 3/2$.

(d) $y\cos x dx + (\sin x - \sin y)dy = 0$. $P = y\cos x$ and $Q = \sin x - \sin y$. Consequently $P_y = \cos x$ and $Q_x = \cos x$ so equation is exact. $F_x = y\cos x$. Thus $F = \int y\cos x dx = y\sin x + h(y)$. Now $F_y = \sin x + h'(y) = \sin x - \sin y$. Hence $h'(y) = -\sin y$. Thus $h(y) = \cos y + C$. So solution is given implicitly by $y\sin x + \cos y + C = 0$.

Question 4.

Homogeneous Equations.

(a) We have the equation $(y^2 - xy)dx + x^2dy = 0$. Which is of course the same as $\dfrac{dy}{dx} = \dfrac{xy - y^2}{x^2}$. Let $y = xv$. Then $y' = v + xv'$ Thus

$$xv' + v = \frac{x^2v - x^2v^2}{x^2} = v - v^2$$

So $xv' = -v^2$. Thus

$$\frac{dv}{-v^2} = \frac{dx}{x}.$$

Integrating gives $\frac{1}{v} = \ln x + C$. Or $v = \frac{1}{\ln x + C}$. Hence $y = \dfrac{x}{\ln x + C}$.

(b) The equation $(x + 3y)dx + xdy = 0$. Naturally this is usually written $\dfrac{dy}{dx} = -1 - \dfrac{3y}{x}$. Setting $y = xv$ leads to $xv' + v = -1 - 3v$ or $v' + \frac{4}{x}v = -\frac{1}{x}$. This if first order linear and the integrating factor is $e^{\int \frac{4}{x}dx} = x^4$. Thus we multiply through by the integrating factor to obtain $(x^4v)' = -x^3$. Integrating gives $x^4v = -\frac{1}{4}x^4 + C$, or $v = -\frac{1}{4} + \frac{C}{x^4}$. The solution is then

$$y = -\frac{1}{4}x + C/x^3.$$

(c) The equation $y' = \frac{x^3}{4x^3 - 3x^2y}$ is homogeneous. Put $y = xv$ again and we find

$$xv' + v = \frac{x^3}{4x^3 - 3x^3v} = \frac{1}{4 - 3v}.$$

Now

$$\begin{aligned} xv' &= \frac{1}{4 - 3v} - v = \frac{1 - v(4 - 3v)}{4 - 3v} \\ &= \frac{1 - 4v + 3v^2}{4 - 3v}. \end{aligned}$$

So that

$$\frac{dv}{dx} = \frac{1 - 4v + 3v^2}{x(4 - 3v)},$$

which is separable. Hence

$$\frac{4 - 3v}{1 - 4v + 3v^2}dv = \frac{dx}{x}.$$

Now we can write

$$\frac{4 - 3v}{1 - 4v + 3v^2} = \frac{A}{v - 1} + \frac{B}{3v - 1}$$

and this leads to

$$\int \frac{4-3v}{1-4v+3v^2}dv = \int \left(\frac{1}{2(v-1)} - \frac{9}{2(3v-1)}\right) dv$$
$$= \frac{1}{2}\ln(v-1) - \frac{3}{2}\ln(3v-1)$$
$$= \frac{1}{2}\ln\left(\frac{v-1}{(3v-1)^3}\right).$$

Hence

$$\frac{1}{2}\ln\left(\frac{v-1}{(3v-1)^3}\right) = \ln x + C.$$

Since $v = y/x$ we have the implicit solution

$$\frac{1}{2}\ln\left(\frac{y/x-1}{(3y/x-1)^3}\right) = \ln x + C.$$

(d) Now $\dfrac{dy}{dx} = \dfrac{x^3y}{x^4+y^4}$. Put $y = xv$ and we have $xv' + v = \dfrac{v}{1+v^4}$. Then

$$xv' = \frac{v}{1+v^4} - v = \frac{v - v(1+v^4)}{1+v^4} = -\frac{v^5}{1+v^4}.$$

Now we have

$$\frac{1+v^4}{v^5}dv = -\frac{dx}{x},$$

and integrating we obtain

$$\ln v - \frac{1}{4v^4} = -\ln x + C.$$

Putting $v = y/x$ gives the implicit solution

$$\ln(y/x) - \frac{1}{4(y/x)^4} = -\ln x + C.$$

(e) Finally we have $e^{y/x}y' = 2(e^{y/x}-1) + \frac{y}{x}e^{y/x}$. The substitution $y = xv$ gives

$$e^v(xv'+v) = 2(e^v-1) + ve^v,$$

hence $xv' + v = v + 2(1-e^{-v})$. Hence $xv' = 2(1-e^{-v})$ or

$$\int \frac{e^v dv}{e^v - 1} = 2\int \frac{dx}{x}. \tag{0.1}$$

Thus $\ln(e^v - 1) = 2\ln x + C$. Hence $e^v - 1 = Ax^2$ where $A = e^C$. Consequently $v = \ln(1 + Ax^2)$. Thus

$$y = x\ln(1+Ax^2).$$