37335 Differential Equations.

Tutorial One Solutions.

Question 1.

First Order Separable Equations.

(a) We have to solve $y' = 2x^2y$ with $y(0) = 2$. The equation is separable. So we can write it as

$$\frac{dy}{y} = 2x^2dx.$$

Integration of both sides gives $\ln y = \frac{2}{3}x^3 + C$. Note that there is a constant of integration on both sides, but we have combined them into a single unknown constant. This is standard practice.

Taking the exponential of both sides leads to $y(x) = Ae^{\frac{2}{3}x^3}$. Since $y(0) = 2 = A$ we have the solution $y(x) = 2e^{\frac{2}{3}x^3}$.

(b) We have $y' = 2xy^2$. This is also separable. We let

$$\frac{dy}{y^2} = 2xdx.$$

Integrating this leads to $\frac{-1}{y} = x^2 + C$. Hence $y = \dfrac{-1}{x^2 + C}$.

(c) This is a little more complicated. $y' = \frac{\cos x}{3y^2+e^y}, y(0) = 2$. We rewrite it as

$$(3y^2 + e^y)dy = \cos xdx.$$

Integration gives $y^3 + e^y = \sin x + C$. Now $y(0) = 2$. So that $(y(0))^3 + e^{y(0)} = \sin 0 + C$. Hence $C = e^2 + 8$. Thus

$$y^3 + e^y = \sin x + 8 + e^2$$

is an *implicit* solution of the equation.

(d) We have $2y' = y(y - 2)$. Separating we have

$$\frac{dy}{y(y-2)} = \frac{1}{2}dx.$$

Partial fractions gives

$$\frac{1}{y(y-2)} = \frac{1}{2}\left(\frac{1}{y-2} - \frac{1}{y}\right).$$

(Check this). Integration then gives

$$\frac{1}{2}\left(\ln(y-2) - \ln y\right) = \frac{1}{2}\ln\left(\frac{y-2}{y}\right) = \frac{1}{2}x + C.$$

So that $\ln\left(\frac{y-2}{y}\right) = x + 2C$. Taking exponentials gives $\frac{y-2}{y} = Ae^x$, $A = e^{2C}$. Hence $y - 2 = yAe^x$. Or $y(1 - Ae^x) = 2$. Finally we have

$$y = \frac{2}{1 - Ae^x}.$$

(e) $2(y-1)y' = e^x, y(0) = -2$. This separates to give

$$2(y-1)dy = e^x dx.$$

Integration gives us the quadratic $y^2 - 2y = e^x + C$. The condition $y(0) = -2$ leads to $4 + 4 = 1 + C$. So $C = 7$. Whence $y^2 - 2y = e^x + 7$. We can use the quadratic formula to obtain

$$y = \frac{2 \pm \sqrt{4 + 4(e^x + 7)}}{2} = 1 \pm \sqrt{e^x + 8}.$$

However we see that if we take the positive square root, the solution does not satisfy the initial condition. So we have

$$y = 1 - \sqrt{e^x + 8}.$$

Question 2.

First Order Linear Equations.

(a) $xy' + 2y = 4x^2, y(1) = 4$. This is the same as $y' + \frac{2}{x} = 4x$. An integrating factor is $e^{\int \frac{2}{x}dx} = e^{\ln x^2} = x^2$. Multiply the equation by the integrating factor to get

$$x^2y' + 2xy = 4x^3.$$

This is $(x^2y) = 4x^3$. Integrate both sides to get $x^2y = x^4 + C$. Or $y = x^2 + \frac{C}{x}^2$. Since $y(1) = 4$ we hqve $4 = 1 + \frac{C}{1}$. Or $C = 3$. So the solution is

$$y = x^2 + \frac{3}{x^2}.$$

(b) We solve $xy' + (x-2)y = 3x^3e^{-x}$. As before we rewrite the equation as $y' + (1 - \frac{2}{x})y = 3x^2e^{-x}$. The integrating factor is

$$e^{\int(1-\frac{2}{x})dx} = e^{x - \ln x^2} = \frac{1}{x^2}e^x.$$

Multiplying both sides of the equation by the integrating factor gives

$$\left(\frac{1}{x^2}e^x y\right)' = 3.$$

So that $\frac{1}{x^2}e^x y = 3x + C$ upon integrating both sides. Hence the solution is

$$y = (3x^3 + Cx^2)e^{-x}.$$

(c) $y'+\cot xy = 3\sin x\cos x$. Recall that $\cot x = \frac{\cos x}{\sin x}$. So that $\int \cot x dx = \int \frac{\cos x}{\sin x}dx = \ln(\sin x)$. So the integrating factor is $e^{\ln(\sin x)} = \sin x$. Multiply both sides of the equation by the integrating factor to get $\sin xy' + \cos xy = (\sin xy)' = 3\sin^2 c\cos x$. Now integrate both sides to get $\sin xy = \int 3\sin^2 x\cos x dx$. For the integral on the right, put $u = \sin x$. Then $du = \cos x$. This gives $\int 3\sin^2 x\cos x dx = 3\int u^2 du = u^3 + C = \sin^3 x + C$. So we have $\sin xy = \sin^3 x + C$. Our solution is thus

$$y = \sin^2 x + \frac{C}{\sin x} = \sin^2 x + C\mathrm{cosec}x.$$

(d) $x\ln xy' + y = 2\ln x, y(e) = 1$. The procedure is the same. Rewrite the equation as $y + \frac{1}{x\ln x}y = \frac{2}{x}$. Now to determine the integrating factor we evaluate $\int \frac{1}{x\ln x}dx = \int \frac{du}{u} = \ln(\ln x)$, where we used the substitution $u = \ln x, dx = \frac{1}{x}dx$ in the integral. Thus $e^{\int \frac{1}{x\ln x}dx} = e^{\ln(\ln x)} = \ln x$. Multiply the equation by the integrating factor to get $((\ln x)y)' = \frac{2\ln x}{x}$. Integration gives

$$(\ln x)y = 2\int \frac{\ln x}{x}dx = \int 2udu = u^2 + C = (\ln x)^2 + C.$$

We used the substitution $u = \ln x, du = \frac{dx}{x}$ again. Our solution is then $y = \ln x + \frac{C}{\ln x}$. Now $y(e) = \ln e + C/\ln e = 1 + C = 1$. Recall $\ln e = 1$. Thus $C = 0$ and the solution satisfying the initial condition is $y = \ln x$.

(e) We solve $y' + 3x^2y = x^2 + e^{-x^3}$. The integrating factor is $e^{\int 3x^2dx} = e^{x^3}$. Multiplying the equation by the integrating factor gives us

$$(e^{x^3}y)' = x^2e^{x^3} + 1.$$

Integrate both sides to produce $e^{x^3}y = \int x^2e^{x^3}dx + x + C = \int \frac{1}{3}e^u du + x + C = \frac{1}{3}e^{x^3} + x + C$ where we put $u = x^3$, $du = 3x^2dx$. Hence the solution is

$$y = \frac{1}{3} + (x + C)e^{-x^3}.$$

Question 3.

Bernoulli Equations.

(a) We solve $y' + \frac{3}{x}y = x^2y^2$. $n = 2$, so put $y^{1-2} = \frac{1}{y} = u$. Then $u' = -\frac{y'}{y^2}$. Divide equation by y^2 to get

$$\frac{y'}{y^2} + \frac{3}{xy} = x^2.$$

This is the same as $-u' + \frac{3}{x}u = x^2$, or $u' - \frac{3}{x}u = -x^2$. This is first order linear and the integrating factor is $e^{-\int \frac{3}{x}dx} = e^{\ln x^{-3}} = \frac{1}{x^3}$. Then we have $\left(\frac{1}{x^3}u\right)' = -\frac{1}{x}$. Integrating gives $\frac{1}{x^3}u = -\ln x + C$. Hence $u = \frac{1}{y} = x^3(C - \ln x)$. Or $y = \frac{1}{x^3(C-\ln x)}$.

(b) $2y' + \frac{1}{x+1}y + 2(x^2-1)y^3 = 0$. Here $n = 3$, so $u = y^{1-3} = \frac{1}{y^2}$. Then $u' = \frac{-2y'}{y^3}$. Divide the equation by y^3 to get

$$\frac{2y'}{y^3} + \frac{1}{y^2(x+1)} = 2(1-x^2).$$

This is

$$-u' + \frac{u}{x+1} = 2(1-x^2).$$

Or $u' - \frac{1}{1+x}u = 2(x^2-1) = 2(x+1)(x-1)$. The integrating factor is

$$e^{-\int \frac{dx}{x+1}} = e^{-\ln(x+1)} = \frac{1}{x+1}.$$

Multiplying by the integrating factor gives us

$$\left(\frac{1}{x+1}u\right)' = 2(x+1).$$

Integrating we find

$$\frac{1}{x+1}u = x^2 + 2x + C.$$

Hence $u = \frac{1}{y^2} = (2x^2 + 2x + C)(x+1)$. Thus we have

$$y^2 = \frac{1}{(2x^2+2x+C)(x+1)}.$$

The branch of the square root that we choose will depend on the initial condition.

(c) $xyy' = y^2 - x^2$. Divide by y. This gives $xy' = y - \frac{x^2}{y}$. Hence $n = -1$. So we put $u = y^{1-(-1)} = y^2$. Then $u' = 2yy'$. Then the equation $xyy' = y^2 - x^2$. becomes $\frac{1}{2}xu' - u = -x^2$. This becomes $u' - \frac{2}{x}u = -2x$. This has integrating factor given by $e^{-\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$. Multiplying by the integrating factor produces

$$\left(x^{-2}u\right)' = -\frac{2}{x}.$$

Integration of both sides gives us $x^{-2}u = -2\ln x + C$, or

$$u = y^2 = x^2(C - 2\ln x).$$