UTS Stochastic Processes and Financial Mathematics (37363)

Assessment Task 2: Assignment

This assignment is marked from 64.

It is worth 25% of the marks for this subject.

This is a group assignment (groups size 2-4), although you may complete individually if you prefer. Please have one individual from each group post group membership details to the "Assignment Groups" thread on the "Discussions" page of the subject Canvas site.

You must use R to answer questions requiring computation in this assignment.

Include images of R code and output you refer to in your answers.

Document your answers showing all necessary steps – if only R code and output is provided then only 1 mark for the question part will be awarded.

Please submit your work (with signed coversheet) as a single PDF file to Canvas.

Due by 23:59 Sunday 1st June 2025.

Question 1 [12 marks].

Let B_t and W_t be standard Brownian motions with $\operatorname{cov}(B_t, W_s) = \frac{9}{10}\min(t, s)$ and define

$$X_t = B_t - W_{t-1}, \quad t \geq 1.$$

(a) Using Theorem 1 from Chapter 5 Notes, determine if X_t is a Markov process **[3 marks]**.

(b) Calculate

$$P(-1 < E[X_4|X_3] < 0).$$

[3 marks]

(c) Calculate

$$P(\operatorname{cov}(X_2, X_2|(X_3, X_1)^T) > 2).$$

[3 marks]

(d) With time step size $\Delta = 1/1000$, simulate a trajectory

$$\left(\hat{X}_j = X_{1+j\Delta}\right)_{j=0}^{1000}$$

and compute $m_1 = \max\left(\hat{X}_0, \hat{X}_1, \dots, \hat{X}_{1000}\right)$. Repeat this $n = 10^5$ times and construct a histogram of $(m_k)_{k=1}^{n}$ **[3 marks]**.

Hint. Use lower-triangular matrix from Cholesky decomposition to simulate correlated normal RVs.

Question 2 [6 marks].

Let $X_t \sim N(0,1)$ and independent for all $t \in \mathbb{Z}$ and consider the process

$$Y_t = \exp(-2X_t + 3X_{t+1}), \qquad t \in \mathbb{Z}.$$

(a) Find $E[Y_t|Y_{t-1}, Y_{t-2}, Y_{t-3}]$ **[3 marks]**.

(b) Determine if Y_t is weakly stationary **[3 marks]**.

Question 3 [10 marks].

Consider the geometric Brownian Motion (GBM)

$$S_t = S_0 \exp\big((r - \sigma^2/2)t + \sigma B_t\big), \qquad 0 \le t \le T,$$

where $S_0 = 15, r = 3/100, \sigma = 1/3, T = 1$ and B_t is a standard Brownian motion.

An example of a discretely-monitored, up-and-in call option with European payoff has price at $t = 0$ given by

$$C = e^{-rT} E\left[\max(S_T - K, 0) I\left(\max_{t\in\tau} S_t > Q\right)\right]$$

where the strike price $K = 10$ and $\tau = \left\{{}^{1}/_{250}, {}^{2}/_{250}, \dots, T\right\}$ is the set of monitoring points for the knock-in barrier $Q = 20$.

Note that the indicator function

$$I\left(\max_{t\in\tau} S_t > Q\right) = \begin{cases} 1, & \max_{t\in\tau} S_t > Q \\ 0, & \max_{t\in\tau} S_t \le Q \end{cases}.$$

(a) Taking $n = 10^5$ paths, use R to calculate the crude Monte Carlo estimate

$$C_n = \frac{e^{-rT}}{n} \sum_{k=1}^{n} \max\left(S_T^{(k)} - K, 0\right) I\left(\max_{t\in\tau} S_t^{(k)} > Q\right)$$

where

$$S_t^{(k)} = S_0 \exp\left((r - \sigma^2/2)t + \sigma B_t^{(k)}\right)$$

and $B_t^{(k)}$ are independent standard Brownian motions. Also calculate a 95% two-sided confidence interval for C **[5 marks]**.

Now consider the European vanilla call option with same parameter values as above and price given by the Black-Scholes formula

$$\begin{aligned} C_{BS} &= e^{-rT} E[\max(S_T - K, 0)] \\ &= S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2) \end{aligned}$$

where

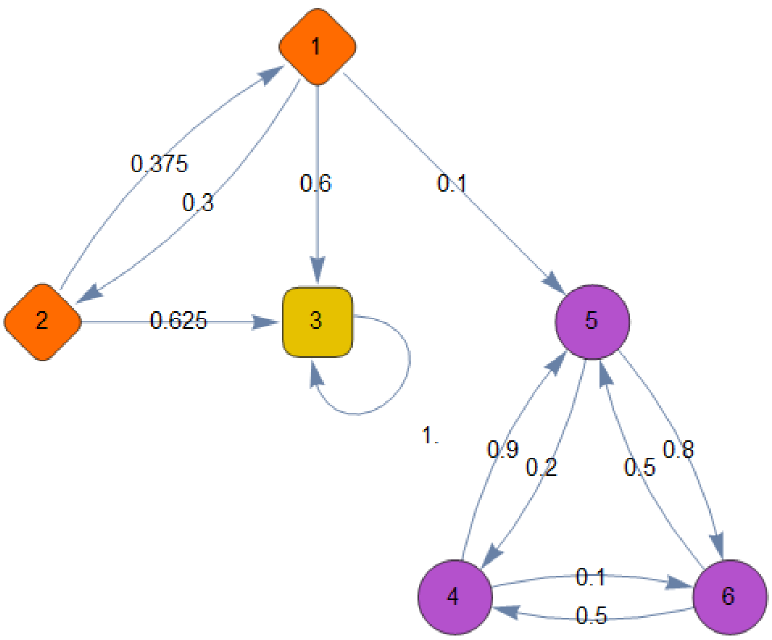
$$d_1 = \frac{1}{\sigma\sqrt{T}} \left(\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T \right), \quad d_2 = d_1 - \sigma\sqrt{T},$$

and Φ is $N(0,1)$ cumulative distribution function.

(b) Again taking $n = 10^5$ paths, calculate a Monte Carlo estimate of C using the discounted vanilla call payoff as a control variate for variance reduction. Also calculate a 95% two-sided confidence interval for C under this scheme **[5 marks]**.

Question 4 [12 marks].

Let X_t, $t \geq 0$, be a continuous time, homogenous Markov chain taking states $X_t \in \{1,2,\dots,6\}$. When jumps occur, assume they happen as depicted in the network diagram below.



Further assume that the average time spent waiting to leave states 1, 2, 4, 5, 6 is 3, 2, 1, 8, 5 respectively and that $X_0 = 1$.

(a) Compute $\mathrm{cov}(X_3, X_{30})$ **[3 marks]**.

(b) Calculate all stationary distributions of X_t and determine if X_t is ergodic **[3 marks]**.

Consider the jump diffusion

$$Y_t = -2t + \frac{1}{2}B_t + \sum_{k=1}^{N_t} H_k, \;\; t \geq 0,$$

where B_t is a standard Brownian motion, N_t is a Poisson process with intensity $1/2$ and $H_k \sim N(-1,1)$ for $k \in \{1,2,\dots,N_t\}$. Assume the SPs and RVs appearing in Y_t are independent of each other.

(c) Suppose $N_7 = 3$. Calculate $P(N_{7+u} = 3, 0 \leq u \leq 4)$ **[3 marks]**.

(d) Determine $E[e^{Y_6} | B_2 = -2, N_4 = 1]$ **[3 marks]**.

Question 5 [12 marks].

Consider the process

$$X_t = -\frac{7}{36}X_{t-1} + \frac{31}{36}X_{t-2} + \frac{1}{3}X_{t-3} + Z_t - \frac{1}{15}Z_{t-1} - \frac{16}{45}Z_{t-2} + \frac{4}{45}Z_{t-3}, \;\; t \in \mathbb{Z},$$

where Z_t is a zero-mean Gaussian white noise process with $\text{var}(Z_t) = 1$.

(a) Identify the process X_t **[3 marks]**.

Now consider the ARMA(2,3) process

$$Y_t = c - \frac{43}{36}Y_{t-1} - \frac{1}{3}Y_{t-2} + Z_t - \frac{1}{15}Z_{t-1} - \frac{16}{45}Z_{t-2} + \frac{4}{45}Z_{t-3}, \;\; t \in \mathbb{Z},$$

where $c \in \mathbb{R}$ and Z_t is a zero-mean Gaussian white noise process with $\text{var}(Z_t) = 1$.

(b) Taking $c = 2$, find $E[Y_3]$ **[3 marks]**.

(c) Taking $c = 0$, determine if Y_t is invertible with respect to Z_t **[3 marks]**.

(d) Taking $c = 0$, compute $\text{cov}(Y_6, Y_8)$ to two decimal places **[3 marks]**.

Question 6 [12 marks].

Consider the process

$$X_t = t + (1 + B_t)^2, \qquad t \geq 0,$$

where B_t is a standard Brownian motion.

(a) Evaluate

$$a(t, x) = \lim_{h \to 0} \frac{E[X_{t+h} - X_t | X_t = x]}{h}$$

to find the drift coefficient of X_t **[3 marks]**.

(b) Find the Ito representation of X_t **[3 marks]**.

Now consider the standard geometric OU process

$$Z_t = \exp\left(e^{-t} + e^{-t} \int_0^t e^u dB_u\right), \qquad t \geq 0,$$

where B_t is a standard Brownian motion.

(c) Show that the transition density function of Z_t is given by

$$f(y, t | x, s) = \frac{1}{y\sqrt{\pi(1 - e^{-2(t-s)})}} \exp\left(-\frac{\left(\log(y) - e^{-(t-s)}\log(x)\right)^2}{1 - e^{-2(t-s)}}\right).$$

[3 marks]

(d) Show that the stochastic differential equation of the process Z_t is given by

$$dZ_t = \left(\frac{1}{2} - \log(Z_t)\right) Z_t dt + Z_t dB_t.$$

[3 marks]