UTS Stochastic Processes and Financial Mathematics (37363)

Sample Exam Autumn 2025

Question 1.

(a) Let B_t be a standard Brownian motion and

$$X_t = e^{-2t+2B_t}, \qquad t \geq 0.$$

Find $\text{corr}(X_t, X_s)$, $0 \leq s < t$.

Now let

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 5 \\ 9 \end{pmatrix} t, \begin{pmatrix} 1 & 2 & -1 \\ 2 & 13 & 8 \\ -1 & 8 & 14 \end{pmatrix} t \right).$$

(b) Find the distribution of $\begin{pmatrix} X_t + 5Y_t \\ 2X_t - 7Y_t \end{pmatrix}$.

(c) Find $E\left[\begin{pmatrix} X_t + 5Y_t \\ 2X_t - 7Y_t \end{pmatrix} | Z_t \right]$.

Question 2.

Consider the Poisson process N_t, $t \geq 0$, with intensity $\lambda = 1$.

(a) Calculate

$$P(N_1 = 4, N_3 = 5 | N_5 = 7).$$

Now define the compound Poisson

$$X_t = \sum_{k=1}^{N_t} Y_k \,, t \geq 0,$$

with N_t as above and where Y_k, $k \in \{1,2, \dots, N_t\}$, has the distribution

$$P(Y_k = i) = \begin{cases} 1/4, & i = 1 \\ 2/5, & i = 2. \\ 7/20, & i = 3 \end{cases}$$

Assume that N_t and the Y_k are all independent.

(b) Find $E\left[e^{iuX_t}\right]$.

(c) Find $\mathrm{cov}(X_4, X_3)$.

Question 3.

Consider the homogenous, discrete-time Markov chain $X_t, t = 0,1,2 \ldots$, taking states $X_t \in \{x_1 = 3, x_2 = -4, x_3 = 2\}$ with one-step transition matrix and initial distribution

$$P(1) = \begin{pmatrix} 2/10 & 3/10 & 5/10 \\ 7/10 & 1/10 & 2/10 \\ 4/10 & 3/10 & 3/10 \end{pmatrix}, \qquad p(0) = \begin{pmatrix} 1/10 \\ 3/10 \\ 6/10 \end{pmatrix}.$$

(a) Calculate $E[X_2]$.

(b) Find a stationary distribution.

(c) Show that the stationary distribution from (b) does not depend on $p(0)$.

Question 4.

Consider the homogenous, continuous-time Markov chain $X_t, t \geq 0$, with transition matrix (when jump occurs)

$$P^{jump} = \begin{pmatrix} 0 & 2/10 & 8/10 \\ 1/2 & 0 & 1/2 \\ 7/10 & 3/10 & 0 \end{pmatrix}.$$

Assume that the distribution of weighting times for jumps from states 1, 2 and 3 have parameters 2, 6 and 7 respectively.

(a) Write down the generator matrix A.

(b) Calculate $P(X_{t+3} = x_2 | X_t = x_1)$.

(c) Find the moment generating function of T_2, where T_2 is the waiting time for a jump from state 2. Make sure to list any conditions(s) necessary for this function to be properly defined.

Question 5.

Consider the ARMA(2,2) process

$$X_t + \frac{1}{4}X_{t-1} - \frac{3}{8}X_{t-2} = Z_t - \frac{13}{35}Z_{t-1} - \frac{12}{35}Z_{t-2}$$

where Z_t is a white noise process with variance σ^2.

(a) Is X_t stationary? Is X_t causal? Is X_t invertible?

(b) What does your answer in (a) mean for reconstructing the current value of the noise process Z_t from X_t?

(c) Find the approximation of the solution to the ARMA(2,2) equation

$$X_t \approx \sum_{j=-3}^{3} \psi_j \left(B^j Z\right)_t.$$

Question 6.

Consider the process

$$X_t = \frac{1}{2}B_t^2 + t, \qquad t \geq 0,$$

where B_t is a standard BM or Wiener process.

(a) Using Definition 1 page 6 Chapter 8 Notes, derive the drift function for X_t.
Hint. Use relationship on page 34 Chapter 8 Notes.

(b) Find the Ito representation of X_t.

(c) Write down the Kolmogorov backward equation for

$$f(y, t|x, s)$$

where

$$f(y, t|x, s) = \frac{\partial}{\partial y} F(y, t|x, s)$$

with

$$F(y, t|x, s) = P(X_t < y | X_s = x).$$

Now let

$$Y_t = e^{X_t}, \quad t \geq 0.$$

(d) Find the Ito representation of Y_t.

(e) Write down the Kolmogorov forward equation for

$$f(y, t|x, s)$$

where

$$f(y, t|x, s) = \frac{\partial}{\partial y} F(y, t|x, s)$$

with

$$F(y, t|x, s) = P(Y_t < y|Y_s = x).$$