Stochastic Processes and Financial Mathematics (37363)

Chapter 11

Elements of diffusion processes, stochastic integration, Ito formula

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Chapter outline

Topics:

- Definitions
    - Approximation of diffusion process
- Transition densities
    - Kolmogorov equations for transition densities
- Kolmogorov backward equation in Black-Scholes model
- Stochastic integrals with respect to Brownian motion
    - Construction of Ito integral

Definitions

In this chapter we discuss the class of Markov processes with continuous trajectories.

This class of stochastic processes occurs frequently in quantitative finance, physics, biology and statistics.

There are two different approaches to define and study this class:

1. **analytical** (via distributions)
2. **trajectory** (via stochastic equations).

Definitions

The **analytical approach** is based on study of finite-dimensional distributions (FDD) which are defined for any $t_i \in D$, $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, as

$$F_{X_{t_1},\ldots,X_{t_n}}(x_1, \ldots, x_n) = P(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n),$$

To describe all FDDs of a **Gaussian process** one needs to know only two functions (mean and covariance functions)

$$m(t) = E[X_t] \quad \text{and} \quad Q(t, s) = \mathrm{cov}(X_t, X_s).$$

To describe all FDDs of a **Markov process** one needs to know only two functions (initial distribution and transition probability function).

Definitions

The **trajectory approach** is based on the study of stochastic relations (equations) which describe the dynamics of a stochastic process.

Examples.
AR(1) process.

$$X_n = \lambda_1 X_{n-1} + \xi_n$$

Geometric Brownian motion.

$$S_t = S_0 \exp(mt + \sigma B_t)$$

Compound Poisson process.

$$X_t = \sum_{n=1}^{N_t} Y_n$$

Definitions

We begin with a definition of a diffusion process using the analytical approach.

Definition 1 (Diffusion process (analytical approach))

A Markov process X_t, $t \in [0, T]$, with continuous trajectories such that for some $\delta > 0$

$$E[|X_t|^{2+\delta}] < \infty$$

is called a diffusion process if there exists functions $a(s, x)$ and $b(s, x)$ such that for $h \to 0$

$$E[X_{s+h} - X_s | X_s = x] = a(s, x)h + o(h) \tag{1}$$

and

$$E\left[(X_{s+h} - X_s)^2 | X_s = x\right] = b^2(s, x)h + o(h). \tag{2}$$

Definitions

Recall that "little o" notation $o(h)$ means that as $h \to 0$

$$o(h)/h = o(1) \to 0.$$

Therefore we can write that as $h \to 0$

$$E[X_{s+h} - X_s | X_s = x]/h \to a(s,x)$$

and

$$E\left[(X_{s+h} - X_s)^2 | X_s = x\right]/h \to b^2(s,x).$$

The function $a(s,x)$ is called a **drift coefficient**

The nonnegative function $b(s,x)$ is called a **diffusion coefficient**.

Example.
A standard Brownian motion B_t, i.e. with $E[B_t] = 0$ and $\text{var}(B_t) = t$, is a diffusion process with continuous trajectories and independent increments.

Hence

$$E[B_{s+h} - B_s | B_s = x] = E[B_{s+h} - B_s] = 0$$

and

$$E\left[(B_{s+h} - B_s)^2 | B_s = x\right] = E\left[(B_{s+h} - B_s)^2\right] = h.$$

Therefore for this case

$$a(s, x) = 0 \quad \text{and} \quad b(s, x) = 1.$$

Actually, as we shall see later on, diffusion processes can be considered as some functions of standard Brownian motions (although these functions may be not explicit).

Definitions

It can be shown that under some general assumptions (e.g. when $a(s,x)$ and $b(s,x)$ are continuous, differentiable functions and $b(s,x) > 0$) that the corresponding diffusion process exists and is unique.

Remarks.
1. To define distributions of a diffusion process we need **three functions**: $a(s,x)$, $b(s,x)$ and a distribution of an initial RV X_0.

2. Existence and uniqueness are understood in the sense of distributions.

3. There exist other definitions of a diffusion process which do not require the assumption $E[|X_t|^{2+\delta}] < \infty$.

For example, see *A Benchmark Approach to Quantitative Finance* [Platen and Heath, 2006].

APPROXIMATION OF DIFFUSION PROCESS.
Let B_t be a standard Brownian motion and assume there exists a process X_t such that X_s and $(B_{s+h} - B_s)$ are independent for any $s > 0$ and

$$X_{s+h} - X_s = a(s, X_s)h + b(s, X_s)(B_{s+h} - B_s) + o(h). \tag{3}$$

We shall show that such an X_t satisfies equations (1) and (2).

First, taking the conditional expectation of (3) gives

$$E[X_{s+h} - X_s | X_s] = a(s, X_s)h + o(h).$$

which is (1).

Definitions

Next note that

$$
\begin{aligned}
(X_{s+h} - X_s)^2 &= a^2(s, X_s)h^2 + b^2(s, X_s)(B_{s+h} - B_s)^2 + (o(h))^2 \\
&\quad + 2a(s, X_s)hb(s, X_s)(B_{s+h} - B_s) + 2a(s, X_s)ho(h) \\
&\quad + 2b(s, X_s)(B_{s+h} - B_s)o(h) \\
&= b^2(s, X_s)(B_{s+h} - B_s)^2 \\
&\quad + 2a(s, X_s)hb(s, X_s)(B_{s+h} - B_s) \\
&\quad + 2b(s, X_s)(B_{s+h} - B_s)o(h) + o(h).
\end{aligned}
$$

Taking the conditional expectation we obtain

$$
E\left[(X_{s+h} - X_s)^2 | X_s\right] = b^2(s, X_s)h + o(h)
$$

which is (2).

So X_t is a diffusion process with drift coefficient $a(s, x)$ and diffusion coefficient $b(s, x)$.

Definitions

The representation (3) can be justified with the use of so-called **stochastic integrals** and **stochastic differential equations** (to be considered later in this chapter).

It can be used for constructing a discrete time approximation $\widehat{X} = (\widehat{X}_{t_i})_{0 \le t_i \le T}$ for the diffusion process $X = (X_t)_{0 \le t \le T}$ by setting

$$h = \frac{T}{n}, \quad t_i = i\frac{T}{n} = ih, \quad i = 0, \dots, n$$

and dropping $o(h)$ terms to give

$$\widehat{X}_{t_i+h} - \widehat{X}_{t_i} = a(t_i, \widehat{X}_{t_i})h + b(t_i, \widehat{X}_{t_i})(B_{t_i+h} - B_{t_i}). \tag{4}$$

Note that increments $B_{t_i+h} - B_{t_i} \sim N(0, h)$ and iid.

Such a recurrent procedure is called "Euler-Maryama approximation" (note there exist other more accurate approximations e.g. "Milstein" approximation etc).

Problem (Brownian motion (BM)).
Consider a Brownian motion W_t with $E[W_t] = mt$ and $\text{var}(W_t) = \sigma^2 t$.

Then $a(s,x) = m$ and $b(s,x) = \sigma$.

Problem (Geometric Brownian motion (gBM)).
Let $S_t = S_0 \exp(mt + \sigma B_t)$ where B_t is a standard Brownian motion.

Show

$$a(s,x) = (m + \sigma^2/2)x \quad \text{and} \quad b(s,x) = \sigma x.$$

Solution. To find the drift coefficient note

$$\begin{aligned} E[S_{s+h} - S_s | S_s = x] &= \left[S_s e^{mh+\sigma(B_{s+h}-B_s)} - S_s | S_s = x\right] \\ &= xE\left[e^{mh+\sigma(B_{s+h}-B_s)} - 1\right] \quad \text{(independent increments)} \\ &= x(e^{mh+\frac{\sigma^2}{2}h} - 1) = x(m + \sigma^2/2)h + o(h). \end{aligned}$$

Definitions

Hence $a(s,x) = (m + \sigma^2/2)x$.

For the diffusion coefficient

$$\begin{aligned}
E\left[(S_{s+h} - S_s)^2 | S_s = x\right] &= \left[(S_s e^{mh+\sigma(B_{s+h}-B_s)} - S_s)^2 | S_s = x\right] \\
&= x^2 E\left[(e^{mh+\sigma(B_{s+h}-B_s)} - 1)^2\right] \quad \text{(independent increments)} \\
&= x^2 E\left[e^{2mh+2\sigma(B_{s+h}-B_s)} - 2e^{mh+\sigma(B_{s+h}-B_s)} + 1\right] \\
&= x^2(e^{2mh+2\sigma^2 h} - 2e^{mh+\sigma^2 h/2} + 1) \\
&= x^2(1 + 2mh + 2\sigma^2 h - 2 - 2mh - \sigma^2 h + 1 + o(h)) \\
&= x^2\sigma^2 h + o(h).
\end{aligned}$$

Hence $b(s,x) = \sigma x$.

Finding drift and diffusion coefficients from stochastic representations can be made much easier with use of Ito formula which will be shown later in the chapter.

Transition densities

REMINDER.
Recall that if X_t is a Markov process then

$$P(X_{t_{n+1}} < x_{n+1} | X_{t_1} = x_1, \ldots, X_{t_n} = x_n) = P(X_{t_{n+1}} < x_{n+1} | X_{t_n} = x_n).$$

Suppose there exists a joint density $f(x_1, t_1; \ldots; x_n, t_n)$, i.e.

$$F_{X_{t_1} \ldots X_{t_n}}(x_1, \ldots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f(u_1, t_1; \ldots; u_n, t_n)\, du_1 \ldots du_n.$$

Denote the transition density function as

$$f(y, t | x, s) := \frac{f(x, s; y, t)}{f(x, s)}.$$

Transition densities

Note that

$$f(y,t) = \int_{-\infty}^{\infty} f(x,s;y,t)dx = \int_{-\infty}^{\infty} f(y,t|x,s)f(x,s)dx.$$

Also recall that for $0 < t_1 < \cdots < t_n \leq T$, $n = 1, 2, \ldots$, that

$$f(x_1,t_1;\ldots;x_n,t_n) = f(x_1,t_1)\prod_{k=2}^{n} f(x_k,t_k|x_{k-1},t_{k-1})$$

and the **Chapman-Kolmogorov equation**

$$f(x_3,t_3|x_1,t_1) = \int_{-\infty}^{\infty} f(x_2,t_2|x_1,t_1)f(x_3,t_3|x_2,t_2)dx_2.$$

holds.

Transition densities

Example (standard Brownian motion).
For $0 \le s < t$

$$f(y, t|x, s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\Big(- \frac{(y-x)^2}{2(t-s)}\Big).$$

To see, note that

$$f(y, t|x, s) = \frac{\partial}{\partial y} P\big(B_t < y | B_s = x\big)$$

where

$$P\big(B_t < y | B_s = x\big) = P\big(B_t - B_s < y - B_s | B_s = x\big) = P\big(B_t - B_s < y - x\big)$$
$$= \int_{-\infty}^{y-x} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{u^2}{2(t-s)}\right) du.$$

Transition densities

Example (geometric Brownian motion).
Let

$$X_t = X_0 e^{mt+\sigma B_t}.$$

where B_t is a standard Brownian motion.

In this case for any $y > 0$, $x > 0$ and $0 \leq s < t$

$$f(y,t|x,s) = \frac{1}{y\sqrt{2\pi\sigma^2(t-s)}}\exp\Big(-\frac{\big(\log\frac{y}{x} - m(t-s)\big)^2}{2\sigma^2(t-s)}\Big).$$

Transition densities

To see this note that

$$f(y,t|x,s) = \frac{\partial}{\partial y} P\big(X_0 e^{mt+\sigma B_t} < y | X_s = x\big)$$

where

$$\begin{aligned}
P\big(X_0 e^{mt+\sigma B_t} < y | X_s = x\big) &= P\big(X_s e^{m(t-s)+\sigma(B_t-B_s)} < y | X_s = x\big) \\
&= P\big(x e^{m(t-s)+\sigma(B_t-B_s)} < y | X_s = x\big) \\
&= P\big(x e^{m(t-s)+\sigma(B_t-B_s)} < y\big) \quad \text{(independent increments)} \\
&= P\big(\sigma(B_t - B_s) < \log(y/x) - m(t-s)\big) \\
&= \int_{-\infty}^{\log(y/x)-m(t-s)} \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \exp\left(-\frac{u^2}{2\sigma^2(t-s)}\right) du.
\end{aligned}$$

Transition densities

Example (standard Ornstein-Uhlenbeck process)
A stationary standard OU process is a Gaussian process with

$$E[X_t] = 0 \quad \text{and} \quad Q(t,s) = \text{cov}(X_t, X_s) = e^{-|t-s|}/2.$$

For any y, x and $t > s$

$$f(y,t|x,s) = \frac{1}{\sqrt{\pi\big(1 - e^{-2(t-s)}\big)}} \exp\left(- \frac{\big(y - xe^{-(t-s)}\big)^2}{\big(1 - e^{-2(t-s)}\big)} \right)$$

Proof. As class work.

Transition densities

Example (geometric Ornstein-Uhlenbeck process)
Let $Y_t = e^{X_t}$ where X_t is a standard Ornstein-Uhlenbeck process.

Then for any $y > 0$, $x > 0$ and $t > s$

$$f(y, t|x, s) = \frac{1}{y\sqrt{\pi\left(1 - e^{-2(t-s)}\right)}} \exp\left(- \frac{\left(\log(y) - \log(x)e^{-(t-s)}\right)^2}{\left(1 - e^{-2(t-s)}\right)} \right).$$

Proof. As class work.

Transition densities

KOLMOGOROV EQUATIONS FOR TRANSITION DENSITIES.

Backward equation.
When y and t are fixed

$$\frac{\partial f(y,t|x,s)}{\partial s} + a(s,x)\frac{\partial f(y,t|x,s))}{\partial x} + \frac{1}{2}b^2(s,x)\frac{\partial^2 f(y,t|x,s)}{\partial x^2} = 0.$$

Forward equation.
When x and s are fixed

$$\frac{\partial f(y,t|x,s)}{\partial t} + \frac{\partial}{\partial y}\big(a(t,y)f(y,t|x,s)\big) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(b^2(t,y)f(y,t|x,s)\big) = 0.$$

The forward equations is also called the Fokker-Planck equation.

Transition densities

Initial conditions.
The backward and forward equations are solved subject to

$$f(s, y|s, x) = \delta(y - x)$$

where Dirac's delta is defined with the property

$$\int_{-\infty}^{\infty} \delta(y - x)dx = 1.$$

Kolmogorov backward equation in Black-Scholes model

Consider a financial market with constant “risk-free” rate r and a gBM model for the stock price

$$S_t = S_0 e^{mt+\sigma B_t}$$

where B_t is a standard Brownian motion

Let the payoff of a European option (i.e. the value of the contract at maturity T) be $g(S_T)$, e.g.

$$g(S_T) = (S_T - K)^+$$

for a European vanilla call.

Then (by the theory of fair pricing of financial contracts) the price of this European option at any moment $s < T$ must be

$$V(x,s) = e^{-r(T-s)} E[g(S_T)|S_s = x].$$

Kolmogorov backward equation in Black-Scholes model

The function $V(x, s)$ can be found as a solution of the backward PDE

$$\frac{\partial V(x,s)}{\partial s} + rx\frac{\partial V(x,s)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V(x,s)}{\partial x^2} = rV(x,s) \qquad (5)$$

for $s < T$ with the boundary condition

$$V(x, T) = g(x).$$

Such PDEs can be solved for $V(x, s)$ numerically using finite difference/element techniques, using certain functions in Mathematica or other computational software packages.

Kolmogorov backward equation in Black-Scholes model

Derivation (outline)
Accordingly to the general theory of pricing of financial contract we need to set $m = r - \sigma^2/2$ so that

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma B_t}.$$

As shown earlier, S_t has drift and diffusion coefficients

$$a(s,x) = (m + \sigma^2/2)x = rx \quad \text{and} \quad b(s,x) = \sigma x$$

respectively.

The fair price of the option at moment s is

$$V(x,s) = e^{-r(T-s)} E[g(S_T)|S_s = x]$$

and this implies the boundary condition because

$$V(x,T) = E[g(S_T)|S_T = x] = g(x).$$

Kolmogorov backward equation in Black-Scholes model

Set for any suitably differentiable function $G = G(s, x)$ the operator

$$L[G] := \frac{\partial G}{\partial s} + rx\frac{\partial G}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 G}{\partial x^2}.$$

Let $f \equiv f(y, T|x, s)$ be the transition density of the process S_t.

We know that f satisfies the Backward Kolmogorov equation

$$L[f] = \frac{\partial f}{\partial s} + rx\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0$$

for $0 \leq s < T$.

Kolmogorov backward equation in Black-Scholes model

Applying the operator L to $V(x,s)$ we obtain

$$
\begin{aligned}
L[V(x,s)] &= L\Big[e^{-r(T-s)}E\big[g(S_T)|S_s=x\big]\Big] \\
&= \frac{\partial}{\partial s}\big(e^{-r(T-s)}E\big[g(S_T)|S_s=x\big]\big) + rx\frac{\partial}{\partial x}\big(e^{-r(T-s)}E\big[g(S_T)|S_s=x\big]\big) \\
&\quad + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}\big(e^{-r(T-s)}E\big[g(S_T)|S_s=x\big]\big) \\
&= re^{-r(T-s)}E\big[g(S_T)|S_s=x\big] + e^{-r(T-s)}\frac{\partial}{\partial s}E\big[g(S_T)|S_s=x\big] \\
&\quad + e^{-r(T-s)}rx\frac{\partial}{\partial x}E\big[g(S_T)|S_s=x\big] + e^{-r(T-s)}\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}E\big[g(S_T)|S_s=x\big] \\
&= rV(x,s) + e^{-r(T-s)}L\Big[E\big[g(S_T)|S_s=x\big]\Big].
\end{aligned}
$$

Kolmogorov backward equation in Black-Scholes model

But

$$L\Big[E\big[g(S_T)|S_s = x\big]\Big] = L\Big[\int_{-\infty}^{\infty} g(y)f(y, T|s, x)dy\Big]$$
$$= \int_{-\infty}^{\infty} g(y)L\big[f(y, T|s, x)\big]dy \quad \text{(can be justified)}$$
$$= 0$$

as

$$L\big[f(y, T|s, x)\big] = 0$$

by the Kolmogorov backward equation.

So

$$L[V(x, s)] = rV(x, s) + e^{-r(T-s)}L\Big[E\big[g(S_T)|S_s = x\big]\Big] = rV(x, s)$$

or

$$\frac{\partial V(x, s)}{\partial s} + rx\frac{\partial V(x, s)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V(x, s)}{\partial x^2} = rV(x, s)$$

which is (5).

Stochastic integrals with respect to Brownian motion

Recall that the diffusion process with the drift coefficient $a(s, x)$ and diffusion coefficient $b(s, x)$ has the following representation

$$X_{s+h} - X_s = a(s, X_s)h + b(s, X_s)(B_{s+h} - B_s) + o(h).$$

After taking the limit $h \to 0$ the stochastic differential equation notation

$$dX_s = a(s, X_s)ds + b(s, X_s)dB_s$$

is employed.

This should be read as shorthand for the (mathematically correct) stochastic integral equation form

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s.$$

Stochastic integrals with respect to Brownian motion

The integral

$$\int_0^t a(s, X_s) ds$$

is a **Lebesgue integral**.

The integral

$$\int_0^t b(s, X_s) dB_s$$

is a **stochastic integral**.

Note that both integrals have a random integrand.

There exist several different constructions (not all equivalent) of stochastic integrals.

We shall always use the most popular construction called the **Ito stochastic integral**.

Stochastic integrals with respect to Brownian motion

TERMINOLOGY.
We say that a stochastic process (random function) f_t **is adapted** to the **filtration** $\mathcal{F}_t$ generated by a standard Brownian motion B_t if for any t the value f_t is completely defined by $\{B_s, s \leq t\}$.

We can interpret the filtration $\mathcal{F}_t$ as the available information (or history) of all possible events involving B_t up to time t, including values of the process itself.

Example.
Adapted functions.

$$f_t = B_{t/2}, \quad f_t = B_t^2, \quad f_t = \int_0^t B_s ds, \quad f_t = \max_{s \leq t} B_s$$

Non-adapted functions.

$$f_t = B_{t+1}^2, \quad f_t = \int_0^{t+10} B_s ds, \quad f_t = \max_{t \leq s \leq t+1} B_s$$

CONSTRUCTION OF ITO INTEGRAL.

Let f_t be an adapted function and consider a uniform partition $t_k = k\frac{t}{n}$, $k = 0, 1, \ldots, n$, and denote

$$Z_n = \sum_{k=1}^{n} f_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) = \sum_{k=1}^{n} f_{t_{k-1}} \Delta B_{t_k}.$$

It can be shown (K. Ito was the first who did this) that under the assumption

$$P\Bigg(\int_0^t f_s^2 ds < \infty \Bigg) = 1$$

there exists a limit $\lim_{n\to\infty} Z_n$ (in probability).

This limit is called a **stochastic integral of** f_t with respect to B_t and is denoted

$$\lim_{n\to\infty} Z_n := \int_0^t f_s dB_s \,.$$

Stochastic integrals with respect to Brownian motion

Problem.
Show that

$$\int_0^t B_s dB_s = \frac{B_t^2 - t}{2}.$$

Solution. Consider a uniform partition $t_k = k\frac{t}{n}$, $k = 0, 1, \ldots, n$, and denote

$$Z_n = \sum_{k=1}^{n} B_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}).$$

As

$$B_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) = \frac{(B_{t_k}^2 - B_{t_{k-1}}^2)}{2} - \frac{(B_{t_k} - B_{t_{k-1}})^2}{2}$$

we have

$$Z_n = \frac{B_t^2}{2} - \frac{1}{2}\sum_{k=1}^{n}(B_{t_k} - B_{t_{k-1}})^2.$$

Stochastic integrals with respect to Brownian motion

We can write

$$(B_{t_k} - B_{t_{k-1}})^2 = \frac{t}{n}\xi_k^2$$

where

$$\xi_k = \frac{B_{t_k} - B_{t_{k-1}}}{\sqrt{t/n}} \sim N(0,1)$$

and independent.

So we have

$$\sum_{k=1}^{n}(B_{t_k} - B_{t_{k-1}})^2 = t\frac{\sum_{k=1}^{n}\xi_k^2}{n} \xrightarrow{P} t$$

as by the Law of Large Numbers

$$\frac{\sum_{k=1}^{n}\xi_k^2}{n} \xrightarrow{P} 1.$$

The desired result follows.

Stochastic integrals with respect to Brownian motion

Proposition 1 (Wiener stochastic integral)

Let $f(t)$ be a continuously differentiable non-random function such that

$$\int_0^t f(s)^2 ds < \infty$$

and let B be a standard BM.

Then

$$\int_0^t f(s) dB_s \sim N\Big(0, \int_0^t f^2(s) ds\Big)$$

and

$$\int_0^t f(s) dB_s = f(t) B_t - \int_0^t B_s \frac{d}{ds} f(s) ds.$$

The last equation is known as N. Wiener's definition of stochastic integrals – Ito later extended this to random integrands.

Stochastic integrals with respect to Brownian motion

The stochastic integral shares properties with more familiar Lebesgue integrals.

Proposition 2 (Properties of stochastic integral)

1.
$$\int_0^t (c_1 g_s + c_2 f_s) dB_s = c_1 \int_0^t g_s dB_s + c_2 \int_0^t f_s dB_s$$

2. for $t \geq s$
$$\int_0^s g_u dB_u + \int_s^t g_u dB_u = \int_0^t g_u dB_u$$

Proof. Omitted.

Stochastic integrals with respect to Brownian motion

Theorem 1 (Expectations of stochastic integral)

1. If $E\left[\sqrt{\int_0^T g_s^2 ds}\right] < \infty$, then for all $t \leq T$

$$E\left[\int_0^t g_s dB_s\right] = 0.$$

2. If $E\left[\int_0^T g_s^2 ds\right] < \infty$, then for all $t \leq T$

$$E\left[\left(\int_0^t g_s dB_s\right)^2\right] = E\left[\int_0^t g_s^2 ds\right] = \int_0^t E\left[g_s^2\right] ds.$$

Note that

$$E\left[\int_0^T g_s^2 ds\right] < \infty \Rightarrow E\left[\sqrt{\int_0^T g_s^2 ds}\right] < \infty.$$

Proof. Omitted.

Stochastic integrals with respect to Brownian motion

Definition 2 (Ito representation 1)

We say that an adapted process X_t has the Ito representation (i.e. is an Ito process) if

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s$$

where a_t and b_t are regular adapted processes such that

$$\int_0^t |a_s| ds < \infty \quad \text{and} \quad \int_0^t b_s^2 ds < \infty.$$

Note that in this case the stochastic integral is the Ito integral introduced earlier.

Examples.

1. Recall

$$B_t^2 = t + 2\int_0^t B_s dB_s$$

which we showed earlier in the chapter.

2. If $f(t)$ is a continuously differentiable non-random function then

$$f(t)B_t = \int_0^t B_s \frac{d}{ds} f(s) ds + \int_0^t f(s) dB_s.$$

Stochastic integrals with respect to Brownian motion

Theorem 2 (Ito formula)

Let a regular adapted process X_t have the Ito representation

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s$$

and a function $g(t,x)$ have continuous derivatives $\frac{\partial}{\partial t} g(t,x)$ and $\frac{\partial^2}{\partial x^2} g(t,x)$. Then the process $g(t,X_t)$ has the Ito representation

$$\begin{aligned} g(t,X_t) = g(0,X_0) \\ &+ \int_0^t \Big(\frac{\partial}{\partial s} g(s,X_s) + a_s \frac{\partial}{\partial x} g(s,X_s) + \frac{1}{2} b_s^2 \frac{\partial^2}{\partial x^2} g(s,X_s) \Big) ds \\ &+ \int_0^t b_s \frac{\partial}{\partial x} g(s,X_s) dB_s. \end{aligned}$$

Stochastic integrals with respect to Brownian motion

This is often written in the shorthand form

$$dg(t, X_t) = \Big(\frac{\partial}{\partial t} g(t, X_t) + a_t \frac{\partial}{\partial x} g(t, X_t) + \frac{1}{2} b_t^2 \frac{\partial^2}{\partial x^2} g(t, X_t)\Big) dt \\ + b_t \frac{\partial}{\partial x} g(t, X_t) dB_t.$$

Also note that the spatial derivative is understood in the sense

$$\frac{\partial}{\partial x} g(t, X_t) \equiv \frac{\partial}{\partial x} g(t, x)|_{x=X_t}.$$

Stochastic integrals with respect to Brownian motion

Proof (outline).
We consider the case $a_t = 0$, $g(t,x) = g(x)$.

Set

$$t_k = k\frac{t}{n} \quad \text{so that} \quad \delta = \frac{t}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

Then

$$X_{t_k} - X_{t_{k-1}} = b_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) + o(\delta)$$

and by Taylor expansion

$$\begin{aligned} g(X_{t_k}) &- g(X_{t_{k-1}}) \\ &= \frac{d}{dx}g(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2}\frac{d^2}{dx^2}g(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2 + o(\delta) \\ &= \frac{d}{dx}g(X_{t_{k-1}})b_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) \\ &\quad + \frac{1}{2}\frac{d^2}{dx^2}g(X_{t_{k-1}})b^2_{t_{k-1}}(B_{t_k} - B_{t_{k-1}})^2 + o(\delta). \end{aligned}$$

Stochastic integrals with respect to Brownian motion

Using $E[B_{t_k} - B_{t_{k-1}}]^2 = \delta$ and summing over the increments gives

$$
\begin{aligned}
g(X_t) - g(X_0) &= \sum_k \frac{d}{dx} g(X_{t_{k-1}}) b_{t_{k-1}} (B_{t_k} - B_{t_{k-1}}) \\
&\quad + \sum_k \frac{1}{2} \frac{d^2}{dx^2} g(X_{t_{k-1}}) b^2_{t_{k-1}} \delta \\
&\quad + \sum_k \frac{1}{2} \frac{d^2}{dx^2} g(X_{t_{k-1}}) b^2_{t_{k-1}} \big((B_{t_k} - B_{t_{k-1}})^2 - \delta \big) + o(1) \\
&= \int_0^t \frac{d}{dx} g(X_s) b_s dB_s + \int_0^t \frac{1}{2} \frac{d^2}{dx^2} g(X_s) b_s^2 ds
\end{aligned}
$$

as $\delta \to 0$ if we can show that

$$
\sum_k \frac{1}{2} g''_{xx}(X_{t_{k-1}}) b^2_{t_{k-1}} \big((B_{t_k} - B_{t_{k-1}})^2 - \delta \big) = o(1).
$$

Stochastic integrals with respect to Brownian motion

Definition 3 (Ito representation 2)

Let X_t have the Ito representation

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s$$

and let f_t be a regular adapted process such that

$$\int_0^t |f_s a_s| ds < \infty \quad \text{and} \quad \int_0^t |f_s b_s|^2 ds < \infty.$$

Then by definition the process $\int_0^t f_s dX_s$ has the representation

$$\int_0^t f_s dX_s = \int_0^t f_s a_s ds + \int_0^t f_s b_s dB_s.$$

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