Stochastic Processes and Financial Mathematics (37363)

Chapter 6

Continuous-time Markov chains, compound Poisson processes

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Autumn 2025

Chapter outline

Topics:

- Continuous-time homogenous Markov chains
  - Birth and death process
  - Kolmogorov equations
  - Limiting probabilities
  - Classification of states
- Compound Poisson processes
  - Processes with independent increments
  - Poisson process
  - Compound Poisson process

Continuous-time homogenous Markov chains

In this section we consider a continuous-time, homogeneous Markov chain X_t with n states and $t \in [0, \infty)$.

Many results from the discrete-time case carry over to the continuous-time case, such as ergodicity and classification of states.

Most importantly, recall from Chapter 5 the Chapman-Kolmogorov equation

$$\boldsymbol{P}(s+u) = \boldsymbol{P}(s)\boldsymbol{P}(u), \quad s, u \geq 0, \tag{1}$$

and state probability vector calculated as

$$\boldsymbol{p}(t+s) = \boldsymbol{P}^T(s)\boldsymbol{p}(t), \quad t, s \geq 0, \tag{2}$$

which are valid for both the discrete and continuous-time cases.

How can we find $\boldsymbol{P}(s)$ and simulate X_t?

Continuous-time homogenous Markov chains

The fundamental difference between discrete and continuous-time Markov chains is that discrete-time chains are not defined between jumps (or state changes) but continuous-time chains are.

Let $X_t = x_i$ and denote the waiting (sojourn) time of state x_i as

$$T_i := \min(s > 0 | X_{t+s} \neq x_i).$$

The waiting times themselves are RVs.

Theorem 1 (waiting times)

Let X_t be a homogeneous, continuous-time Markov chain.

Then all waiting times T_i are independent and have an exponential distribution with some parameter $\nu_i \geq 0$, i.e. $T_i \sim \text{Exp}(\nu_i)$.

The ν_i are called the "intensities" or "instantaneous rates" of the chain.

Proof.
By definition of T_i we have

$$\{T_i > s\} = \{X_{t+u} = x_i, 0 \leq u \leq s\}$$

so

$$\begin{aligned}
P(T_i > v + s | T_i > v) &= P(X_{t+u} = x_i, 0 \leq u \leq v + s | X_{t+u} = x_i, 0 \leq u \leq v) \\
&= P(X_{t+u} = x_i, v \leq u \leq v + s | X_{t+u} = x_i, 0 \leq u \leq v) \\
&\text{(because of condition)} \\
&= P(X_{t+u} = x_i, v \leq u \leq v + s | X_{t+v} = x_i) \quad \text{(Markov property)} \\
&= P(X_{t+u} = x_i, 0 \leq u \leq s | X_t = x_i) \quad \text{(homogeneity)} \\
&= P(X_{t+u} = x_i, 0 \leq u \leq s) \\
&= P(T_i > s).
\end{aligned}$$

Continuous-time homogenous Markov chains

That is, we have the memoryless property

$$P(T_i > v + s | T_i > v) = P(T_i > s),$$

a property only possessed by the exponential distribution (not proved).

So

$$\begin{aligned} P(T_i > s) &= 1 - P(T_i \leq s) \\ &= 1 - (1 - e^{-\nu_i s}) \\ &\text{(CDF of } T_i \sim \text{Exp}(\nu_i)) \\ &= e^{-\nu_i s} \end{aligned}$$

for $0 \leq \nu_1 < \infty$.

Continuous-time homogenous Markov chains

In order to obtain the **transition probability matrix**

$$\boldsymbol{P}(s) = \big(p_{ij}(s)\big)_{1\le i,j\le n} = \begin{pmatrix} p_{11}(s) & p_{12}(s) & \cdots & p_{1n}(s) \\ p_{21}(s) & p_{22}(s) & \cdots & p_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(s) & p_{n2}(s) & \cdots & p_{nn}(s) \end{pmatrix} \tag{3}$$

where

$$p_{ij}(s) = P(X_{t+s} = x_j | X_t = x_i), \quad i, j \in \{1, \ldots, n\},$$

we first consider the **jump probability matrix**

$$\boldsymbol{P}^{\text{jump}} = \big(p_{ij}\big)_{1\le i,j\le n} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \tag{4}$$

where

$$p_{ij} = P(X = x_j \text{ after jumping from } X = x_i), \quad i, j \in \{1, \ldots, n\},$$

Continuous-time homogenous Markov chains

The jump probability matrix $\boldsymbol{P}^{\text{jump}}$ contains the probabilities that apply **when a jump occurs**, and can be considered analogous to the 1-step transition probability matrix of a discrete-time chain. Note this matrix does not depend s.

The transition probability matrix $\boldsymbol{P}(s)$ incorporates the information from $\boldsymbol{P}^{\text{jump}}$ but also takes into account the (exponential) waiting times between jumps. Note that this matrix does depend on s.

Of course, the usual properties

$$0 \leq p_{ij}(s), p_{ij} \leq 1, \quad \sum_j p_{ij}(s) = 1 \quad \text{and} \quad \sum_j p_{ij} = 1$$

apply.

If state x_i is an **absorbing state**, then $p_{ii} = 1$, $p_{ij} = 0$ for $i \neq j$ and $\nu_i = 0$.

If state x_i is **not an absorbing state**, then $p_{ii} = 0$ and $\nu_i > 0$.

Continuous-time homogenous Markov chains

Define the **instantaneous transition rate**

$$a_{ij} := \nu_i p_{ij} \quad \text{for} \quad i \neq j \quad \text{and} \quad a_{ii} = -\nu_i.$$

For **non-absorbing states** x_i

$$\sum_{j\neq i} a_{ij} = \sum_{j\neq i} \nu_i p_{ij} = \nu_i \sum_{j\neq i} p_{ij} = \nu_i$$

and

$$p_{ij} = \frac{a_{ij}}{\nu_i} = \frac{a_{ij}}{\sum_{j\neq i} a_{ij}}, \;\; \nu_i \neq 0.$$

We also see

$$\sum_{j} a_{ij} = \sum_{j\neq i} a_{ij} + a_{ii} = \nu_i - \nu_i = 0$$

So, if we are given a_{ij} we can find ν_i and p_{ij} (or vice versa).

Continuous-time homogenous Markov chains

We collect this information together into the **generator matrix** or **intensity matrix**

$$\boldsymbol{A} = \left(a_{ij}\right)_{1\le i,j\le n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} -\nu_1 & \nu_1 p_{12} & \cdots & \nu_1 p_{1n} \\ \nu_2 p_{21} & -\nu_2 & \cdots & \nu_2 p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n p_{n1} & \nu_n p_{n2} & \cdots & -\nu_n \end{pmatrix}. \qquad (5)$$

For **absorbing states** x_i set $\nu_i = 0$.

Example (credit rating).
Let the possible ratings (states) be $x_1 = AA$, $x_2 = B$, $x_3 = C$, $x_4 = D$ (default) with generator matrix

$$\boldsymbol{A} = \begin{pmatrix} -0.06 & 0.03 & 0.02 & 0.01 \\ 0.1 & -0.4 & 0.2 & 0.1 \\ 0.2 & 0.4 & -1.0 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

with the rates (intensities) given per year.

Noting that state $x_4 = D$ is absorbing, it follows that

$$\boldsymbol{P}^{\text{jump}} = \begin{pmatrix} 0 & \frac{a_{12}}{\nu_1} & \frac{a_{13}}{\nu_1} & \frac{a_{14}}{\nu_1} \\ \frac{a_{21}}{\nu_2} & 0 & \frac{a_{23}}{\nu_2} & \frac{a_{24}}{\nu_2} \\ \frac{a_{31}}{\nu_3} & \frac{a_{32}}{\nu_3} & 0 & \frac{a_{41}}{\nu_3} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3/6 & 2/6 & 1/6 \\ 1/4 & 0 & 2/4 & 1/4 \\ 1/5 & 2/5 & 0 & 2/5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Continuous-time homogenous Markov chains

BIRTH AND DEATH PROCESSES.
Consider the population of current size $X_t = i$ where the time until next arrival A_i is exponentially distributed with parameter λ_i and the time until next departure D_i is exponentially distributed with parameter μ_i.

Also suppose the RVs A_i and D_i are independent.

X_t a called a "birth and death process", where the parameters λ_i and μ_i are called the birth and death rates respectively.

So X_t is a continuous-time Markov chain with states $x_i = i$, $i = 0, 1, \ldots$, with sojourn times

$$T_i = \min(A_i, D_i).$$

If $A_i > D_i$ then X_t enters state $i-1$, i.e. $X_{t+D_i} = i - 1$ $(i \geq 1)$.

If $A_i \leq D_i$ then X_t enters state $i+1$, i.e. $X_{t+A_i} = i + 1$ $(i \geq 0)$.

Continuous-time homogenous Markov chains

Examples.
The "Poisson process" has parameters

$$\lambda_i = \lambda > 0, \quad \mu_i = 0, \quad i \geq 0$$

and so has a constant arrival rate and no departures.

The "Yule process" has parameters

$$\lambda_i = i\lambda, \quad \mu_i = 0, \quad i \geq 1$$

and so has a linear arrival rate and no departures.

The process with parameters

$$\lambda_i = i\lambda + \theta, \quad i \geq 0 \text{ and } \mu_i = i\mu, \quad i \geq 1$$

has linear arrival and departure rates.

Continuous-time homogenous Markov chains

Theorem 2 (birth and death process)

A birth and death process is a continuous-time homogeneous Markov chain with exponentially distributed sojourn times $T_i \sim \text{Exp}(\nu_i)$ having the rates

$$\nu_i = \begin{cases} \lambda_0 & \text{if } i = 0 \\ \lambda_i + \mu_i & \text{if } i \geq 1 \end{cases}$$

and the jump probabilities

$$p_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i \geq 1 \quad \text{and} \quad p_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad i \geq 0.$$

Proof.

Step 1. First we show that $T_i \sim \text{Exp}(\nu_i)$. Note that for $i = 0$

$$P(T_0 > s) = P(A_0 > s)$$

and so $T_0 \sim A_0 \sim \text{Exp}(\lambda_0)$.

Continuous-time homogenous Markov chains

Next observe that for $i = 1, 2, \ldots$

$$\begin{aligned}
P(T_i > s) &= P\big(\min(A_i, D_i) > s\big) = P(A_i > s, D_i > s) \\
&= P(A_i > s)P(D_i > s) \quad \text{(independence)} \\
&= \int_s^\infty \lambda_i e^{-\lambda_i x} dx \int_s^\infty \mu_i e^{-\mu_i x} dx \\
&\text{(using PDF of exponential RVs)} \\
&= e^{-\lambda_i s} e^{-\mu_i s} = e^{-(\lambda_i + \mu_i)s} \\
&= \int_s^\infty (\lambda_i + \mu_i) e^{-(\lambda_i + \mu_i)x} dx
\end{aligned}$$

and we recognise the PDF of an exponential RV, so $T_i \sim \text{Exp}(\nu_i)$ with

$$\nu_i = \begin{cases} \lambda_0 & \text{if } i = 0 \\ \lambda_i + \mu_i & \text{if } i \geq 1 \end{cases}.$$

Continuous-time homogenous Markov chains

Step 2. Next we derive the jump probabilities. We have

$$P(A_i \le a, D_i \le d) = P(A_i \le a)P(D_i \le d) \quad \text{(independence)}$$
$$= \int_0^a \lambda_i e^{-\lambda_i x} dx \int_0^d \mu_i e^{-\mu_i x} dx = \int_0^a \int_0^d \lambda_i e^{-\lambda_i x} \mu_i e^{-\mu_i y} dy dx$$

so

$$p_{i,i-1} = P(D_i < A_i) = P(0 < A_i < \infty, 0 \le D_i < A_i)$$
$$= \int_0^\infty \int_0^x \lambda_i e^{-\lambda_i x} \mu_i e^{-\mu_i y} dy dx = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i \ge 1,$$

and

$$p_{i,i+1} = P(A_i \le D_i) = 1 - P(D_i < A_i)$$
$$= 1 - p_{i,i-1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad i \ge 1.$$

To complete the proof we note that (obviously) $p_{0,1} = 1$.

Continuous-time homogenous Markov chains

KOLMOGOROV EQUATIONS.
So, now we know how to find the probabilities p_{ij} that apply when a jump occurs and therefore have $\boldsymbol{P}^{\text{jump}}$ introduced in (4).

But how do find the probabilities $p_{ij}(s)$ that also incorporate the waiting times and therefore obtain $\boldsymbol{P}(s)$ introduced in (3)?

We do so by solving Kolmogorov's differential equations.

These come in two flavours: **backward** and **forward**.

Continuous-time homogenous Markov chains

Theorem 3 (Kolmogorov's differential equations)

Backward Equations.

$$\frac{d}{ds} p_{ij}(s) = \sum_{k \neq i} a_{ik} p_{kj}(s) + a_{ii} p_{ij}(s)$$

or

$$\frac{d}{ds} \boldsymbol{P}(s) = \boldsymbol{A}\boldsymbol{P}(s).$$

Forward Equations.
Under some regularity conditions

$$\frac{d}{ds} p_{ij}(s) = \sum_{k \neq j} p_{ik}(s) a_{kj} + a_{jj} p_{ij}(s)$$

or

$$\frac{d}{ds} \boldsymbol{P}(s) = \boldsymbol{P}(s)\boldsymbol{A}.$$

Continuous-time homogenous Markov chains

Proof.
Technical so omitted.

A solution of the backward Kolmogorov equations for the case of homogenous Markov chains with a finite number of states can be written in the explicit form

$$\boldsymbol{P}(s) = e^{s\boldsymbol{A}} = \boldsymbol{I} + \sum_{k=1}^{\infty} \frac{s^k \boldsymbol{A}^k}{k!} = \left(\boldsymbol{I} + \sum_{k=1}^{\infty} \frac{\boldsymbol{A}^k}{k!} \right)^s \tag{6}$$

where $\boldsymbol{I}$ is the identity matrix.

Then the Chapman-Kolmogorov equation (1) becomes

$$\boldsymbol{P}(s+u) = \boldsymbol{P}(s)\boldsymbol{P}(u) = e^{(s+u)\boldsymbol{A}}, \quad t, u \geq 0,$$

and state probability (2) vector becomes

$$\boldsymbol{p}(t+s) = \boldsymbol{P}^T(s)\boldsymbol{p}(t) = e^{s\boldsymbol{A}^T}\boldsymbol{p}(t), \quad t, s \geq 0,$$

Continuous-time homogenous Markov chains

Example (credit rating continued).
Recall the generator matrix

$$\boldsymbol{A} = \begin{pmatrix} -0.06 & 0.03 & 0.02 & 0.01 \\ 0.1 & -0.4 & 0.2 & 0.1 \\ 0.2 & 0.4 & -1.0 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}.$$

The Mathematica example below computes $\boldsymbol{P}(s)$ for various s.

```
AA = {{-0.06, 0.03, 0.02, 0.01}, {0.1, -0.4, 0.2, 0.1}, {0.2, 0.4, -1.0, 0.4},
   {0, 0, 0, 0}};
PP[s_] := MatrixExp[s AA];
Table[PP[s] // Chop // MatrixForm, {s, {0, 100, 1000}}]
```

$$\left\{ \begin{pmatrix} 1. & 0 & 0 & 0 \\ 0 & 1. & 0 & 0 \\ 0 & 0 & 1. & 0 \\ 0 & 0 & 0 & 1. \end{pmatrix}, \begin{pmatrix} 0.0238099 & 0.00325194 & 0.00116929 & 0.971769 \\ 0.0120147 & 0.00164096 & 0.000590034 & 0.985754 \\ 0.00993054 & 0.0013563 & 0.000487682 & 0.988225 \\ 0 & 0 & 0 & 1. \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1. \\ 0 & 0 & 0 & 1. \\ 0 & 0 & 0 & 1. \\ 0 & 0 & 0 & 1. \end{pmatrix} \right\}$$

So as $s \to \infty$ the probability of default goes to one!

Continuous-time homogenous Markov chains

Example (interest rate model with two states).
Assume the interest rate process X_t takes only the two states $x_1 = 0.05$ and $x_2 = 0.06$ with the generator

$$\boldsymbol{A} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

with rates given per year (e.g. $\lambda = \mu = 5$).

The waiting times between jumps are exponentially distributed, so, for example, the mean waiting time for the jump $x_1 \to x_2$ is $\frac{1}{\lambda}$ while the variance of the waiting time for the jump $x_2 \to x_1$ is $\frac{1}{\mu^2}$.

The backward equations are

$$\frac{d}{ds} p_{1,1}(s) = -\lambda p_{1,1}(s) + \lambda p_{2,1}(s), \quad p_{1,2}(s) = 1 - p_{1,1}(s)$$

and

$$\frac{d}{ds} p_{2,1}(s) = \mu p_{1,1}(s) - \mu p_{2,1}(s), \quad p_{2,2}(s) = 1 - p_{2,1}(s).$$

Continuous-time homogenous Markov chains

One can check that the solutions are

$$p_{1,1}(s) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)s} + \frac{\mu}{\lambda+\mu}$$

and

$$p_{2,1}(s) = \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)s}).$$

It follows that for large s

$$p_{1,1}(s) \approx \frac{\mu}{\lambda+\mu}, \quad p_{1,2}(s) = 1 - p_{11}(s) \approx \frac{\lambda}{\lambda+\mu}$$

and

$$p_{2,1}(s) \approx \frac{\mu}{\lambda+\mu}, \quad p_{2,2}(s) = 1 - p_{21}(s) \approx \frac{\lambda}{\lambda+\mu}.$$

LIMITING PROBABILITIES.

Theorem 4 (ergodic continuous-time Markov chain)

A homogenous Markov chain X_t is called ergodic if $\lim_{t\to\infty} \boldsymbol{p}(t)$ exists and this limit does not depend on the initial distribution $\boldsymbol{p}(0)$.

If a Markov chain is ergodic, then for all initial states i

$$\lim_{s\to\infty} p_{ij}(s) = \lim_{t\to\infty} p_j(t) = \pi_j \quad \text{or} \qquad \lim_{t\to\infty} \boldsymbol{p}(t) = \boldsymbol{\pi}$$

so that

$$\lim_{t\to\infty} \boldsymbol{p}(t+s) = \lim_{t\to\infty} \left(\boldsymbol{P}^T(s)\boldsymbol{p}(t)\right)$$

becomes

$$\boldsymbol{\pi} = \boldsymbol{P}^T(s)\boldsymbol{\pi} = e^{s\boldsymbol{A}^T}\boldsymbol{\pi} = (\boldsymbol{I} + \frac{s\boldsymbol{A}^T}{1!} + \frac{s^2(\boldsymbol{A}^T)^2}{2!} + \cdots)\boldsymbol{\pi}$$

where $\boldsymbol{\pi}$ is the stationary distribution of X_t.

That is, the stationary distribution can be found as the solution to

$$\boldsymbol{A}^T \boldsymbol{\pi} = \mathbf{0} \quad \text{subject to} \quad \sum_{j=1}^{n} \pi_j = 1. \tag{7}$$

CLASSIFICATION OF STATES.
This is the same as for the discrete-time case, with the exception that periodicity does not apply to the continuous-time case (refer to Chapter 5).

Compound Poisson processes

PROCESSES WITH INDEPENDENT INCREMENTS.
We begin with a general definition and proposition.

Definition 1 (process with independent increments)

A stochastic process X_t, $t \in D$, is said to possess independent increments if any RVs $X_t - X_s$ and $X_u - X_0$ are independent for any $u \leq s \leq t$.

Note that it is typically supposed that $X_0 = 0$.

Proposition 1 (process with independent increments is Markov)

If a stochastic process X_t has independent increments then it is a Markov process.

Compound Poisson processes

Proof.
For $X_0 = 0$ and any $y, x_1, \ldots, x_n, x$

$$
\begin{aligned}
&P(X_{t+s} \leq y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \\
&\quad = P(X_{t+s} - X_t \leq y - x | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \\
&\quad = P(X_{t+s} - X_t \leq y - x \\
&\qquad\qquad | X_{t_1} - X_0 = x_1 - 0, X_{t_2} - X_{t_1} = x_2 - x_1 \ldots, X_t - X_{t_n} = x - x_n) \\
&\quad = P(X_{t+s} - X_t \leq y - x) \quad \text{(independent increments)} \\
&\quad = P(X_{t+s} - X_t \leq y - x | X_t = x) \\
&\quad = P(X_{t+s} - X_t + X_t \leq y - x + x | X_t = x) \\
&\quad = P(X_{t+s} \leq y | X_t = x)
\end{aligned}
$$

which is the Markov property.

Compound Poisson processes

Note that if a process with independent increments X_t has a discrete distribution then

$$
\begin{aligned}
&P(X_{t_1} = x_1, X_{t_2} = x_2, \ldots, X_{t_n} = x_n) \\
&\quad = P(X_{t_1} = x_1, X_{t_2} - X_{t_1} = x_2 - x_1 \ldots, X_{t_n} - X_{t_{n-1}} = x_n - x_{n-1}) \\
&\quad = P(X_{t_1} = x_1) \prod_{k=2}^{n} P(X_{t_k} - X_{t_{k-1}} = x_k - x_{k-1}).
\end{aligned}
$$

So, all n-dimensional distributions can be expressed in terms of only one function, the distribution of increments.

Similarly, for the case of continuous distributions we have

$$
f_{X_{t_1},\ldots,X_{t_n}}(x_1, \ldots, x_n) = f_{X_{t_1}}(x_1) \prod_{k=2}^{n} f_{X_{t_k} - X_{x_{k-1}}}(x_k - x_{k-1}).
$$

Compound Poisson processes

Definition 2 (stationary and independent increments)

A stochastic process X_t is said to possess stationary independent increments if any increments $X_t - X_s$ for disjoint intervals (s, t) are independent and the distribution of $X_t - X_s$ is dependent only on the interval $t - s$.

Example (Wiener process).
The increments of the Wiener process

$$(W_t - W_s) \sim N\big(m(t-s), \sigma^2(t-s)\big)$$

and the transition density is

$$f(y, t|x, s) = \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} e^{-\frac{(y-x-m(t-s))^2}{2\sigma^2(t-s)}}.$$

Compound Poisson processes

POISSON PROCESS.

Definition 3 (Counting process)

A stochastic process N_t, $t \in [0, \infty)$, is said to be a counting process (or, a point process) if N_t represents the number of "events" that have occurred up to time t.

Properties of N_t include

1. N_t is integer-valued;
2. if $s \leq t$ then $N_s \leq N_t$;
3. for $s < t$, $N_t - N_s$ is the number of events occurring in $[s, t]$.

Examples of applications include

- number of telephone calls received
- number of stock trades etc
- number of jumps in a stochastic process.

Compound Poisson processes

Denote by S_n the arrival time of the n-th event and let T_n be the interarrival times, i.e. the time taken between the $(n-1)$-th and n-th events, so that

$$S_n = \sum_{i=1}^{n} T_i.$$

We provide three definitions of a Poisson process.

Definition 4 (Poisson process 1)

The counting process N_t with $N_0 = 0$ is said to be a Poisson process with intensity (or rate) $\lambda > 0$ if N_t has independent increments and for all $s, t \geq 0$

$$P(N_{t+s} - N_t = k) = e^{-\lambda s} \frac{(\lambda s)^k}{k!}, \quad k = 0, 1, \ldots.$$

Compound Poisson processes

Definition 5 (Poisson process 2)

The counting process N_t with $N_0 = 0$ is said to be a Poisson process with intensity (or rate) $\lambda > 0$ if N_t is a Markov process with

$$P(N_{t+h} - N_t = 1) = \lambda h + o(h),$$
$$P(N_{t+h} - N_t \geq 2) = o(h)$$

as $h \to 0$.

Definition 6 (Poisson process 3)

The counting process N_t with $N_0 = 0$ is said to be a Poisson process with intensity (or rate) $\lambda > 0$ if the interarrival times $T_n \sim \text{Exp}(\lambda)$ and independent.

Theorem 5 (equivalence of Poisson process definitions)

Definitions 4, 5 and 6 are equivalent.

Proof.
Omitted.

For our purposes, Definitions 4 and 6 will be the most useful.

The next proposition establishes some fundamental properties of a Poisson process.

Proposition 2 (properties of Poisson process)

If N_t is a Poisson process with rate $\lambda > 0$ then the CF

$$\varphi_{N_t}(u) := E[e^{iuN_t}] = \exp(\lambda t(e^{iu} - 1))$$

and the MGF

$$M_{N_t}(u) := E[e^{uN_t}] = \exp(\lambda t(e^{u} - 1)).$$

Also,

$$E[N_t] = \mathrm{var}(N_t) = \lambda t \quad \text{and} \quad \mathrm{cov}(N_t, N_s) = \lambda \min(t, s).$$

Compound Poisson processes

Proof.
The proofs of the forms of φ_{N_t} and M_{N_t} are class exercises.

The CF or MGF can be used to derive the mean and variance formulae.

For the covariance property note that for $s < t$

$$\begin{aligned}
\mathrm{cov}(N_t, N_s) &= \mathrm{cov}(N_t - N_s + N_s, N_s) \\
&= \mathrm{cov}(N_t - N_s, N_s) + \mathrm{cov}(N_s, N_s) \\
&= \mathrm{cov}(N_s, N_s) \quad \text{(independent increments)} \\
&= \mathrm{var}(N_s) \\
&= \lambda s.
\end{aligned}$$

By similar reasoning for $t < s$ we have $\mathrm{cov}(N_t, N_s) = \lambda t$.

Combining the two cases gives the desired result.

Compound Poisson processes

The final property we establish concerns the sum of independent Poisson processes.

Proposition 3 (sum of independent Poisson processes)

Let $N_t^{(1)}$ and $N_t^{(2)}$ be independent Poisson processes with rates λ_1 and λ_2.

Then $Z_t = N_t^{(1)} + N_t^{(2)}$ is the Poisson process with rate $\lambda_1 + \lambda_2$.

Proof.

By the moment generating function of Proposition 4

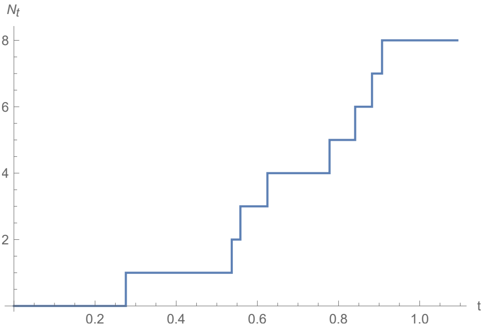
$$\begin{aligned} E[e^{uZ_t}] &= E\left[e^{u(N_t^{(1)}+N_t^{(2)})}\right] = E\left[e^{uN_t^{(1)}}\right]E\left[e^{uN_t^{(2)}}\right] \quad \text{(independence)} \\ &= e^{\lambda_1 t(e^u-1)}e^{\lambda_2 t(e^u-1)} \\ &= e^{(\lambda_1+\lambda_2)t(e^u-1)} \end{aligned}$$

which is the MGF of Poisson process with rate $\lambda_1 + \lambda_2$.

Compound Poisson processes

The Mathematica example below shows a simulated path of a Poisson process N_t with $\lambda = 10$.

```
λ = 10; TT = 1;
data = RandomFunction[PoissonProcess[λ], {0, TT}];
ListStepPlot[data, AxesLabel → {"t", "Nₜ"}]
```



Compound Poisson processes

COMPOUND POISSON PROCESS.

Definition 7 (Compound Poisson process)

A stochastic process X_t is said to be a compound Poisson process if it can be represented as

$$X_t = \sum_{k=1}^{N_t} Y_k$$

where N_t is a Poisson process with rate λ and Y_k are iid RVs which are also independent of N_t.

Example (unit jump size).
Let $X_0 = 0$ and $Y_k \equiv 1$.

Then $X_t = N_t$.

Compound Poisson processes

Example (insurance company model).
Let RV Y_k be claim size.

If the interarrival times T_n are independent and have the exponential distribution with parameter $\lambda > 0$, then the total loss is

$$X_t = \sum_{k=1}^{N_t} Y_k.$$

The insurance company will be ruined when the loss X_t exceeds the equity of the company for the first time.

Compound Poisson processes

The next theorem provides the characteristic function a compound Poisson process.

Theorem 6 (characteristic function of compound Poisson process)

A compound Poisson Process X_t has independent increments and its characteristic function is given by

$$\varphi_{X_t}(u) := E[e^{iuX_t}] = e^{\lambda t(\varphi_Y(u)-1)}$$

where $\varphi_Y(u) := E[e^{iuY_k}]$.

Proof.

The independence of increments property follows the independence of increments of a Poisson process and the independence of the Y_k RVs.

Compound Poisson processes

The characteristic function

$$
\begin{aligned}
E[e^{iuX_t}] &= E\left[e^{iu\sum_{k=1}^{N_t} Y_k}\right] = E\left[E\left[e^{iu\sum_{k=1}^{N_t} Y_k} | N_t\right]\right] \quad \text{(tower law)} \\
&= \sum_{n=0}^{\infty} E\left[e^{iu\sum_{k=1}^{N_t} Y_k} | N_t = n\right] P(N_t = n) \\
&= \sum_{n=0}^{\infty} E\left[e^{iu\sum_{k=1}^{n} Y_k}\right] P(N_t = n) \quad \text{(condition, independence)} \\
&= \sum_{n=0}^{\infty} \prod_{k=1}^{n} E\left[e^{iuY_k}\right] P(N_t = n) \quad \text{(independence)} \\
&= \sum_{n=0}^{\infty} E\left[e^{iuY_k}\right]^n P(N_t = n) \quad (Y_k \text{ iid}) \\
&= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\left(\lambda t \varphi_Y(u)\right)^n}{n!} = e^{-\lambda t} e^{\lambda t \varphi_Y(u)} \quad \text{(Taylor series)} \\
&= e^{\lambda t(\varphi_Y(u) - 1)}.
\end{aligned}
$$

Compound Poisson processes

The next corollary establishes some properties of a compound Poisson process.

Corollary 1 (properties of compound Poisson process)

If $E(Y^2) < \infty$ then

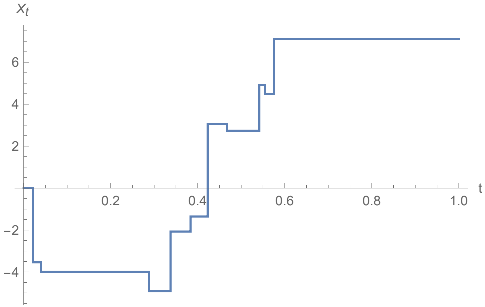
$$E[X_t] = \lambda t E[Y] \quad \text{and} \quad \text{var}(X_t) = \lambda t E[Y^2].$$

Proof.
As class exercises.

Compound Poisson processes

The Mathematica example below shows a simulated path of a compound Poisson process X_t with $\lambda = 10$ and $Y_k \sim N(1, 4)$.

```
λ = 10; TT = 1; μ = 1; σ = 2;
data = RandomFunction[
   CompoundPoissonProcess[λ, NormalDistribution[μ, σ]],
   {0, TT}];
ListLinePlot[data, InterpolationOrder → 0, AxesLabel → {"t", "Xt"}]
```



Compound Poisson processes

Example.
Suppose families migrate to Australia at a Poisson rate of $\lambda = 15$ (per day) and that each family has $1, 2, 3$ and 4 members with probabilities $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{6}$. What is the mean and variance of the number of migrants arriving per week?

Solution. Suppose additionally that families migrate independently. Then the number of migrants

$$X_t = \sum_{k=1}^{N_t} Y_k$$

is the compound Poisson Process with $E(Y_k) = \frac{5}{2}$ and $E(Y_k^2) = \frac{43}{6}$.

Therefore

$$E[X_7] = 15 \times 7 \times \frac{5}{2} = 262\frac{1}{2},$$
$$\mathrm{var}(X_7) = 15 \times 7 \times \frac{43}{6} = 752\frac{1}{2}.$$

References I