UTS

Stochastic Processes and Financial Mathematics

Lab/Tutorial 11

This lab/tutorial is assessed and marked from 18.

To obtain maximum possible marks show all steps necessary to derive answers and explain your reasoning where necessary. If only final line of answer is provided then only 1/3 marks will be awarded for the question part.

Unless otherwise stated, you may use R (or Mathematica) for calculations, but code and output does not constitute an answer. If only this is provided then 1/3 marks will be awarded for the question part.

Include images of all code/output you refer to in your answers. If relevant code/output is not provided, then only 1 out of 3 possible marks will be awarded for the question part irrespective of the answer given.

Please write up your answers to these questions and upload your work in PDF format to Canvas.

Due by 23:59 Tuesday 13th May 2025.

Question 1 [9 marks]

An Ornstein-Uhlenbeck process X_t, $t \geq 0$, can be represented in a variety of ways, one being

$$X_t = X_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma e^{-\theta t} \int_0^t e^{\theta u} dB_u$$

where $\mu \in \mathbb{R}$ and $\sigma, \theta > 0$ and B_t is a standard Brownian Motion.

(a) For $0 < s < t$, show that

$$\int_s^t e^{\theta u} dB_u \sim N\left(0, \frac{e^{2t\theta} - e^{2s\theta}}{2\theta}\right)$$

[3 marks]

Hint 1. Use Proposition 1 (page 36 Chapter 8 Notes) but note that this result is for integrals of the form

$$\int_0^t f(u) dB_u .$$

Defining Y_t as the OU process with $Y_0 \in \mathbb{R}$ and $\mu = 0$ and $\sigma = \theta = 1$ we have

$$Y_t = Y_0 e^{-t} + e^{-t} \int_0^t e^u dB_u$$

which can be shown to also have the representation

$$Y_t = e^{-(t-s)} Y_s + e^{-t} \int_s^t e^u dB_u . \quad (A)$$

(b) Derive the transition density function of Y_t **[3 marks]**.

Hint 2. Use form (A).

(c) Derive $\mathrm{cov}(Y_t, Y_s)$ for $0 \leq s < t$ **[3 marks]**.

Question 2 [9 marks]

Consider the GBM

$$X_t = \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)$$

where $t \geq 0, \sigma > 0$ and B_t is a standard Brownian motion.

(a) Show that X_t has Ito representation

$$X_t = 1 + \int_0^t rX_u du + \int_0^t \sigma X_u dB_u.$$

[not assessed]

Hint 1. Using the Ito representation

$$B_t = \int_0^t 1 dB_u,$$

Take

$$g(t, B_t) = \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)$$

and apply Ito formula (see Theorem 2 page 41 Chapter 8 Notes).

(b) Using the Ito representation in (a), we have

$$\begin{aligned}
X_{t+\Delta t} - X_t &= 1 + \int_0^{t+\Delta t} rX_u du + \int_0^{t+\Delta t} \sigma X_u dB_u - \left(1 + \int_0^t rX_u du + \int_0^t \sigma X_u dB_u\right) \\
&= \int_t^{t+\Delta t} rX_u du + \int_t^{t+\Delta t} \sigma X_u dB_u \\
&\approx \int_t^{t+\Delta t} rX_t du + \int_t^{t+\Delta t} \sigma X_t dB_u \\
&= rX_t \int_t^{t+\Delta t} du + \sigma X_t \int_t^{t+\Delta t} dB_u \\
&= rX_t \Delta t + \sigma X_t (B_{t+\Delta t} - B_t) \\
&\stackrel{d}{=} rX_t \Delta t + \sigma X_t B_{\Delta t}.
\end{aligned}$$

This gives the Euler scheme approximate recursion formula

$$\hat{X}_{t+\Delta t} = \hat{X}_t + r\hat{X}_t \Delta t + \sigma \hat{X}_t B_{\Delta t}.$$

We also know from previous work that we have the exact recursion formula

$$X_{t+\Delta t} = X_t \exp\left((r - \sigma^2/2)\Delta t + \sigma B_{\Delta t}\right).$$

Taking $r = 1/25, \sigma = 1/2, T = 3$ and $\Delta t = T/1000$, use R to simulate a path of X_t and $\hat{X}_t$ and plot the difference $X_t - \hat{X}_t$ **[3 marks]**.

(c) Find the Ito representation of

$$Y_t = \ln(t + X_t).$$

[3 marks]

Hint 2. Using the Ito representation for X_t given in (a), take

$$g(t, X_t) = \ln(t + X_t)$$

and apply Ito formula (see Theorem 2 page 41 Chapter 8 Notes).

(d) Write down the Kolmogorov forward equation that applies to the transition density function of Y_t **[3 marks]**.

(e) Consider the process

$$X_t = \alpha \frac{T-t}{T} + \beta \frac{t}{T} + (T-t) \int_0^t \frac{1}{T-u} dB_u$$

where $0 \leq t < T, \beta \in \mathbb{R}$ and B_t is a standard Brownian motion. Using the Ito formula, show that X_t satisfies the stochastic integral equation

$$X_t = \alpha + \int_0^t \frac{\beta - X_u}{T-u} du + B_t.$$

[not assessed]

Hint 3. Use the Ito representation

$$Z_t = \int_0^t \frac{1}{T-u} dB_u$$

so that

$$X_t = \alpha \frac{T-t}{T} + \beta \frac{t}{T} + (T-t) Z_t.$$

Hint 4. Take

$$g(t, Z_t) = \alpha \frac{T-t}{T} + \beta \frac{t}{T} + (T-t) Z_t$$

and apply Ito formula (see Theorem 2 page 41 Chapter 8 Notes).