UTS

Stochastic Processes and Financial Mathematics

Lab/Tutorial 12

This lab is not assessed.

Question 1

Consider Spencer's filter

$$Y_t = \sum_{j=-7}^{7} \psi_j X_{t-j}$$

with

$$\{\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\} = \{74, 67, 46, 21, 3, -5, -6, -3\}/320,$$
$$\psi_{-j} = \psi_j, \qquad j \neq 0.$$

(a) Show that this filter, when applied to a process with a cubic polynomial trend component, leaves the drift unchanged.

Hint 1. Apply the filter to

$$X_t = m_t + Z_t\,, \qquad t \in \mathbb{Z},$$

where

$$m_t = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

and use Mathematica to perform calculations.

(b) Using R and setting $c_0 = -20$, $c_1 = 4$, $c_2 = -0.13$, $c_3 = 0.001$ and $Z_t \sim N(0,10)$, simulate a path of X_t for $t \in \{1,2,\ldots,100\}$. Plot m_t, X_t and Y_t for $t \in \{1,2,\ldots,100\}$, leaving the values of Y_t undefined for $t \in \{1,,\ldots,7\} \cup \{94,,\ldots,100\}$.

Question 2

(a) Consider the AR(1) process

$$X_t = \lambda X_{t-1} + Z_t$$
$$= \sum_{j=0}^{\infty} \lambda^j Z_{t-j}, \qquad t \in \mathbb{Z},$$

with $|\lambda| < 1$. Show that the process is causal and find

$$\operatorname{cov}\left(\frac{X_1 + X_3 + X_5}{3}, \frac{X_2 + X_4}{2}\right)$$

where $\operatorname{var}(Z_t) = \sigma^2$.

(b) Consider the MA(q) process

$$X_t = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j}, \qquad t \in \mathbb{Z}.$$

Find $\operatorname{cov}(X_t, X_{t+h})$ where $\operatorname{var}(Z_t) = \sigma^2$ and $h = 0,1,2, \ldots..$

Hint 1. Consider 3 cases: $h = 0$, $0 < h \leq q$ and $h > q$.

(c) Consider the ARMA(2,2) process

$$X_t = -\frac{11}{21} X_{t-1} + \frac{2}{7} X_{t-2} + Z_t - \frac{6}{5} Z_{t-1} - \frac{8}{5} Z_{t-2}, \qquad t \in \mathbb{Z}.$$

Find the approximation of the solution to the ARMA(2,2) equation

$$X_t \approx \sum_{j=-4}^{4} \psi_j \left(B^j Z\right)_t.$$

Question 3

(a) Recall the standard OU process

$$Y_t = Y_0 e^{-t} + e^{-t} \int_0^t e^u dB_u.$$

Find the Ito representation of Y_t.

(b) Using R, plot an Euler scheme approximation of Y_t over the interval $0 \leq t \leq 10$ with time step $\Delta t = 0.01$ taking $Y_0 = 0$.

(c) Consider the Ito process

$$X_t = 1 + \int_0^t rX_u du + \int_0^t \sigma X_u dB_u.$$

Find the Ito representation of

$$Z_t = \ln X_t - t.$$

Note 1. There is no problem taking the log as the GBM $X_t > 0$.

(d) Using R, plot an Euler scheme approximation of Z_t over the interval $0 \leq t \leq 10$ with time step $\Delta t = 0.01$ taking $r = 0.04$ and $\sigma = 0.5$.