UTS

Stochastic Processes and Financial Mathematics (37363)

Lab/Tutorial 1

This lab is not assessed.

In this week's lab we review the mathematics necessary for successful completion of this course.

Q1. Calculus

(a) Consider the random variable $X \sim \text{Exp}(\lambda)$ with density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\lambda > 0$.

By performing appropriate integration, show that $\text{var}(X) = 1/\lambda^2$.

Hint. Use result from page 29 Chapter 1 notes and $\text{var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$.

Replicate this result by using the Mathematica function `Integrate`.
Replicate this result by using the Mathematica function `Variance`.
Replicate this result by using the Mathematica function `Expectation`.

(b) Consider the bivariate random variable

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \text{N}\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix} \right)$$

with density function

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right) \right)$$

where $\sigma_X, \sigma_Y > 0$ and $-1 < \rho < 1$.

By performing appropriate integration, show that the (marginal) density function for X is given by

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right).$$

Hint. Using change of variables

$$u = \frac{x-\mu_X}{\sigma_X}, \qquad v = \frac{y-\mu_Y}{\sigma_Y}$$

and completing the square technique, solve

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

Replicate this result by using the Mathematica function `Integrate` (you will need to specify the bounds on σ_X, σ_Y and ρ using `Assumptions` or `Assuming`).

(c) Using the density function f_X, show that $E[X] = \mu_X$.

Hint. Consider $\frac{d}{dx}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$ and the fact that density functions integrate to 1.

(d) The characteristic function of the random variable $X \sim N(\mu_X, \sigma_X^2)$ is given by

$$\varphi_X(\omega) = \exp\left(i\mu_X\omega - \frac{1}{2}\sigma_X^2\omega^2\right)$$

where $\sigma_X > 0$.

By performing appropriate differentiation, show that the second moment

$$\mathrm{E}[X^2] = \mu^2 + \sigma^2.$$

Hint. See page 32 Chapter 1 notes.

Replicate this result by using the Mathematica function `D`.
Replicate this result by using the Mathematica function `Expectation`.

(e) The density function of X in (b) can be recovered from the characteristic function of X in (c) by performing the inverse Fourier transform

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\,\varphi_X(\omega)d\omega,$$

but this expression is well-defined only if

$$\int_{-\infty}^{\infty} \left|e^{-i\omega x}\varphi_X(\omega)\right| d\omega < \infty.$$

Show this last statement is true.

Hint. Recall Euler's formula $e^{iuX} = \cos(uX) + i\,\sin(uX)$.

Recover $f_X(x)$ by performing the inverse Fourier transform using the Mathematica function `InverseFourierTransform` (you will need to set `FourierParameters` when using this function).

Q2. Ordinary differential equations

(a) Using the separation of variables method, find the general solution of

$$x^2\frac{dy}{dx} + x(x+2)y = 0$$

where $x, y > 0$.

Replicate this result by using the Mathematica function `DSolve`.

(b) Using the integrating factor method, find the general solution of

$$x^2\frac{dy}{dx} + x(x+2)y = e^x$$

where $x > 0$.

Hint. The integrating factor is $e^{\int\left(1+\frac{2}{x}\right)dx}$.

Replicate this result by using the Mathematica function `DSolve`.

Q3. Linear algebra

(a) Presenting the solution in vector form, solve the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 2 & 1 & 11 \\ 4 & 3 & 25 \\ 0 & 1 & 3 \end{bmatrix}, b = \begin{bmatrix} 18 \\ 40 \\ 4 \end{bmatrix}.$$

Replicate this result by using the Mathematica function `LinearSolve`.

(b) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}.$$

Replicate this result by using the Mathematica function `Eigensystem`.

Q4. Taylor series

Consider the function $f(x) = \cos(x)\, e^{2x}$.

(a) Write out the first three terms in the Taylor series for $f(x)$ about the point $x = a$. Replicate this result by using the Mathematica function `Series`.

(b) Plot $f(x)$ and the Taylor series approximation for $a \in \{-1,0,1\}$.