UTS

Stochastic Processes and Financial Mathematics (37363)

Lab/Tutorial 2

This lab is not assessed.

Question. The classical probability model

Consider a fair coin (equal probability of heads or tail) and the set of events

$$E_k = \{k \text{ heads from } n \text{ throws}\}.$$

(a) Derive an approximation for the probability of throwing exactly $k = n/4$ heads, i.e. an approximation for $P(E_{n/4})$.

Hint 1. If $X \sim B(n, p = 1/2)$ then $P(E_k) = P(X = k)$. Then by Chapter 1 Notes (page 21)

$$P(X = k) = \binom{n}{k} p^k (1-p)^{(n-k)} = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{(n-k)}.$$

Hint 2. Stirling's approximation $n! = \sqrt{2\pi n} n^n e^{-n}(1 + o(1))$ as $n \to \infty$ can be used when n is large. Substitute $n! = \sqrt{2\pi n} n^n e^{-n}$ into the probability mass function for binomial RV.

"Little *o*" notation. The statement $f(n) = o(1)$ as $n \to \infty$ means that for any $c > 0$ there exists an $n_0 > 0$ such that $|f(n)| < c$ for $n > n_0$.

(b) Using Mathematica, calculate the relative accuracy of this approximation (i.e. ratio of approximation to true value) for the cases $n = 10^2$ and $n = 10^4$.

Hint 1. Use the approximation from (a) divided by the true probability given by the probability mass function for binomial RV above.

(c) Estimate the time required to compute the true value of $P(E_{n/4})$ when $n = 10^9$.

Note. Don't use Mathematica functions `BinomialDistribution` or `Binomial` as these are implemented very efficiently and will defeat the purpose of the task. Instead calculate the factorials with `Factorial` or `!`.

Hint 1. If $t(n)$ is the computational time needed to evaluate $P(E_{n/4})$, then $t(n) \approx Cn^{\alpha}$, where the constants C and α can be evaluated from the equations

$$t(n_i) \approx Cn_i^{\alpha}, \qquad i = 1,2.$$

Hint 2. Take $n_1 = 10^5$ and then calculate the time $t(n_1)$ taken to compute the true value of $P(E_{n_1/4})$ using the Mathematica Function `Timing`. Repeat for $n_2 = 10^6$ to obtain $t(n_2)$.

Hint 3. Solve the approximation $t(n_i) \approx Cn_i^{\alpha}$ for α and for C using the values n_1, n_2, $t(n_1)$ and $t(n_2)$ from Hint 2 to obtain these constants.

Hint 4. Using the values of α and C found in Hint 3, compute the approximation $t(10^9)$.

Question 2. Conditional expectation for discrete RVs

Consider the discrete random variables X, Y, Z with joint probability mass function (PMF) $p_{X,Y,Z}$ described by

$$\begin{array}{ll} p_{X,Y,Z}(1,1,1) = {}^{5}/_{32} & p_{X,Y,Z}(2,1,1) = {}^{4}/_{32} \\ p_{X,Y,Z}(1,1,2) = {}^{3}/_{32} & p_{X,Y,Z}(2,1,2) = {}^{2}/_{32} \\ p_{X,Y,Z}(1,2,1) = {}^{7}/_{32} & p_{X,Y,Z}(2,2,1) = {}^{6}/_{32} \\ p_{X,Y,Z}(1,2,2) = {}^{2}/_{32} & p_{X,Y,Z}(2,2,2) = {}^{3}/_{32} \end{array}$$

(a) Calculate $E[YZ]$.

Hint 1. Using Definition 13 of Chapter 1 Notes (page 39) calculate

$$E[YZ] = \sum_{y}\sum_{z} yz * P(Y = y, Z = z).$$

(b) Calculate $E[X|Y = 2]$.

Hint 1. Using Definition 18 of Chapter 1 Notes (page 46) calculate

$$E[X|Y = 2] = \sum_x x * P(X = x|Y = 2).$$

(c) Calculate $E[YZ|X = 1]$.

(d) Calculate $E[X|Y = 2, Z = 2]$.

Question 3. Conditional expectation for continuous RVs

Consider the continuous random variables X, Y with joint probability density function $f_{X,Y}$ given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{7}{54} x e^{-7y/x - x/3}, & x, y > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Derive the conditional density, $f_{Y|X}$, of $Y|X = x$.

Hint 1. Find the marginal density of X, f_X, by performing the integration

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

and then use Definition 19 of Chapter 1 Notes (page 47).

(b) Show that $f_{Y|X}$ is indeed a density function.

Hint 1. Use the fundamental integration property of a density function.

(c) Identify the distribution of $Y|X = x$.

Hint 1. Compare conditional density found in (a) to examples from Chapter 1 Notes.

(d) Calculate $E[Y|X = x]$.

Hint 1. Use Definition 19 of Chapter 1 Notes (page 47).

(e) Replicate these results from (a), (b) and (d) using the Mathematica function `Integrate`.

Hint 1. Refer to previous lab/tutorial.