UTS

Stochastic Processes and Financial Mathematics (37363)

Lab/Tutorial 3

This lab/tutorial is assessed and marked from 30.

To obtain maximum possible marks show all steps necessary to derive answers and explain your reasoning where necessary. If only final line of answer is provided then only 1/3 marks will be awarded for the question part.

Unless otherwise stated, you may use Mathematica (or the like) for calculations, but Mathematica code and output does not constitute an answer. If only this is provided then 1/3 marks will be awarded for the question part.

Please write up your answers to these questions and upload your work in PDF format to Canvas.

Due by 23:59 Sunday 9th March 2025.

Question 1. Properties of log-normal distribution

Let $X_k \sim LN(\mu_k, \sigma_k^2)$ with $\mu_k \in \mathbb{R}$ and $\sigma_k > 0$ for $k = 1,2$ (i.e. log-normally distributed with parameters μ_k and σ_k). Suppose further that X_1 and X_2 are independent.

Now consider the random variable

$$Y = \frac{\sqrt{X_1}}{X_2^2}.$$

(a) Determine the distribution of Y **[3 marks]**.

Hint 1. As $X_k \sim LN(\mu_k, \sigma_k^2)$ we have $\ln X_k \sim N(\mu_k, \sigma_k^2)$. So

$$\ln Y = \ln \frac{\sqrt{X_1}}{X_2^2} = ?$$

is a linear-affine transformation of the joint-Gaussian random vector

$$\ln X = \begin{pmatrix} \ln X_1 \\ \ln X_2 \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right).$$

Hint 2. Use Theorem 2 of Chapter 2 Notes to determine distribution of $\ln Y$ (why not Definition 1?).

Hint 3. The exponential of a normal RV is log-normal with the same parameters.

(b) Without using computational software (and without integrating), determine $\mathrm{var}(Y)$ **[3 marks]**.

(c) Write down the probability density function f_Y of the random variable Y **[not assessed]**.

Hint 4. Use Mathematica functions `PDF` and `LogNormalDistribution` with parameters found from Hint 3.

(d) Using the R function `integrate` calculate $\mathrm{var}(Y)$ where $\mu_1 = \mu_2 = \sigma_1 = \sigma_2 = 1$ **[3 marks]**.

(e) Repeat the calculations from (d) using the Mathematica function `Variance` **[not assessed]**.

(f) Repeat the calculations from (d) using the Mathematica function `Expectation` **[not assessed]**.

Question 2. Properties of gamma distribution

Let $X_1 \sim \mathrm{Gamma}(2\alpha, 3\beta)$, $X_2 \sim \mathrm{Gamma}(\alpha/3, \beta/5)$ (i.e. gamma-distributed with parameters $2\alpha, 3\beta$ and $\alpha/3, \beta/5$ respectively) and independent. Consider the random variable

$$Y = \frac{1}{3}X_1 + 5X_2.$$

(a) Find the characteristic function ψ_Y of the random variable Y **[3 marks]**.

Hint 1. The characteristic function ψ_Z of the random variable $Z \sim \mathrm{Gamma}(\alpha, \beta)$ is

$$\psi_Z(u) = (1 - iu\beta)^{-\alpha}.$$

(b) From ψ_Y determine the distribution of Y **[3 marks]**.

Hint 2. Compare the result for ψ_Y to ψ_Z in Hint 1.

(c) Using the Mathematica function `D`, calculate $E[Y]$ and $\mathrm{var}(Y)$ where $\alpha = \beta = 1$ **[not assessed]**.

Hint 3. Use moment generating function.

(d) Using the R function `integrate`, calculate $\mathrm{var}(Y)$ where $\alpha = \beta = 1$ **[3 marks]**.

(e) Repeat these calculations in (d) using the Mathematica function `Expectation` **[not assessed]**.

Question 3. Theorem on normal correlation

Let $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 3 \end{bmatrix}\right)$.

Now consider the RVs

$$H_1 = -3X + 4Y \quad \text{and} \quad H_2 = -X - 2Y + 7Z.$$

(a) Determine $E[H_1]$, $\mathrm{var}(H_2)$ and $\mathrm{cov}(H_1, H_2)$ **[3 marks]**.

(b) Find $V = E[H_1|Z]$ using TNC **[3 marks]**.

Hint 1. For RVs A_i, B_j and constants a_i, b_j

$$\mathrm{cov}\left(\sum_{i=1}^{m} a_i A_i, \sum_{j=1}^{n} b_j B_j\right) = \sum_{i=1}^{m} \sum_{i=1}^{n} a_i b_j \,\mathrm{cov}(A_i, B_j).$$

(c) Find $E[(H_1 - V)^2]$ using TNC **[3 marks]**.

(d) Find $E[(H_1 - V)^2]$ without using TNC **[3 marks]**.

Hint 2. Use the relationship

$$E[(2X - 6Z - V)^2] = \mathrm{var}(2X - 6Z - V) + E[2X - 6Z - V]^2.$$