UTS

Stochastic Processes and Financial Mathematics (37363)

Lab/Tutorial 4

This lab is not assessed.

Question 1. Central limit theorem

Consider the "chi-squared" RV with $n \in \mathbb{N}_{>0}$ degrees of freedom

$$Y_n = \sum_{k=1}^{n} X_k^2,$$

where $X_k \sim N(0,1)$ and independent. To denote this we write $Y_n \sim \chi^2(n)$.

(a) Find the MGF of X_k^2.

$$M_{X_k^2}(u) = E\left[e^{uX_k^2}\right], \qquad u < \frac{1}{2}.$$

(b) Show that the MGF of Y_n is given by

$$M_{Y_n}(u) = E[e^{uY_n}] = (1-2u)^{-n/2}, \qquad u < \frac{1}{2}.$$

Hint 1. Use the result from (a) and the independence property for the RVs X_k.

(c) Show that

$$Z_n = \frac{Y_n - n}{8\sqrt{3n}} \xrightarrow{d} N\left(0, \frac{1}{96}\right)$$

as $n \to \infty$. Here 1/96 is variance.

Hint 2. Use the CLT along with $E[X_k^2] = 1$ and $\mathrm{var}(X_k^2) = 2$.

(d) Calculate the relative accuracy (i.e. ratio of estimate to true) of

$$P\left(Y_n < \frac{n^2+1}{n}\right),$$

where the approximate distribution Y_n can be obtained using the results from (c). Do this for $n = 10^4$ and $n = 10^5$.

Hint 3. Use the Mathematica functions `CDF`, `NormalDistribution` and `ChiSquareDistribution` or R functions `pnorm` and `pchisq`.

Question 2. Monte Carlo evaluation of integrals

Consider the integral

$$G = \int_1^8 3x\mathrm{e}^{-x/2} dx.$$

(a) Write the integral in the form

$$G = \int_{-\infty}^{\infty} g(x) f_X(x) dx = E[g(X)]$$

where f_X is the density of $X \sim U(1,8)$.

(b) Find $G = E[g(X)]$.

(c) Find $\sigma^2(g) = \mathrm{var}\big(g(X)\big)$.

(d) Using Mathematica and taking $n = 10^5$, calculate the Monte Carlo estimate

$$G_n = \frac{1}{n} \sum_{k=1}^{n} g(X_k)$$

and standard deviation (i.e. standard error) of the Monte Carlo estimate.

Hint 1. Use the Mathematica functions `RandomReal` and `UniformDistribution`. Also, see Chapter 3 Notes for example code.

(e) Repeat part (d) using R.

Hint 2. Use the R functions `runif` and `sapply`.

(f) Using the results from (c) and the approximation

$$\frac{G_n - G}{\sqrt{\mathrm{var}(G_n)}} \sim N(0,1),$$

calculate a 95% two-sided confidence interval for G.

Hint 3: use the results from (b)-(c) and the Mathematica functions `InverseCDF` and `NormalDistribution`.

(g) Repeat part (f) using R.

Hint 4: use R function `qnorm`.

Extra Work.
Repeat Q2 using by simulating appropriate $X \sim \mathrm{Exp}(\lambda)$ RVs.