UTS

Stochastic Processes and Financial Mathematics (37363)

Lab/Tutorial 5

This lab is not assessed.

Question 1. MC with variance reduction

Consider again the integral from last week

$$G = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx = 18e^{-1/2} - 60e^{-4}$$

where

$$g(x) = 21xe^{-x/2}$$

and

$$f_X(x) = \frac{1}{7} I(1 \leq x \leq 8)$$

is the density of $X \sim U(1,8)$,

(a) Using the control variate $q(x) = \sqrt{x}$ and $n = 10^5$, use R to calculate the Monte Carlo estimate

$$\tilde{G}_n = G_n - a^*(Q_n - Q)$$

where

$$G_n = \frac{1}{n} \sum_{k=1}^{n} g(X_k),$$

$$Q_n = \frac{1}{n} \sum_{k=1}^{n} q(X_k),$$

$$Q = E[q(X)].$$

In calculating a^* to minimise $\operatorname{var}(\tilde{G}_n)$ use sample statistics. Also, calculate the sample estimate of the standard error $\sqrt{\operatorname{var}(\tilde{G}_n)}$.

Hint 1. To calculate a^* use the sample estimates of $\operatorname{var}(q(X))$ and $\operatorname{cov}(g(X), q(X))$.

(b) Using the antithetic variate $h(X) = g(2E[X] - X)$ and $n = 10^5$, use R to calculate the Monte Carlo estimate

$$\hat{G}_n = \frac{1}{2n}\sum_{k=1}^{n}\big(g(X_k) + h(X_k)\big).$$

Also calculate the sample estimate of $\sqrt{\text{var}\big(\hat{G}_n\big)}$.

(c) Without using Mathematica or the like, show that $(2E[X] - X) \sim U(1,8)$ where $X \sim U(1,8)$.

Hint 2. Use characteristic function/moment generating function.

Question 2. Generating RVs

Let the RV X have the density function

$$f_X(x) = \frac{24}{(3x+2)^3}, \qquad x \geq 0.$$

(a) Find the inverse distribution function F_X^{-1} of X.

Hint 1. Find the distribution function

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du.$$

Hint 2. When deriving inverse distribution function F_X^{-1} you will find two candidate solutions, but only one of these satisfies $x \geq 0$.

(b) Using the inverse transformation method and the results from (a), use R to generate a sample $X_k \sim X$, $k \in \{1,2,\dots,10^5\}$, construct a histogram of these simulated RVs and over the top of the histogram superimpose the PDF f_X.

Hint 3. Use the R functions `runif` and `sapply` to generate sample, R function `hist` to construct the histogram and R function `curve` to superimpose the PDF.

Question 3. TNC for Gaussian process

Let X_t, $t \geq 0$, be a Gaussian process with $E[X_t] = -t^2$ and covariance function

$$Q(s,t) = \operatorname{cov}(X_s, X_t) = e^{-4|t-s|}\cos\left(\frac{\pi}{2}|t-s|\right)$$

for $0 \leq s \leq t$.

(a) Find the density and characteristic function of X_2.

(b) Find $E\left[\begin{pmatrix} X_3 \\ X_5 \end{pmatrix} | X_1\right]$.

(c) Find the conditional autocovariance

$$\operatorname{cov}\left(\begin{pmatrix} X_3 \\ X_5 \end{pmatrix}, \begin{pmatrix} X_3 \\ X_5 \end{pmatrix} | X_1\right) = E\left[\left(\begin{pmatrix} X_3 \\ X_5 \end{pmatrix} - E\left[\begin{pmatrix} X_3 \\ X_5 \end{pmatrix} | X_1\right]\right)\left(\begin{pmatrix} X_3 \\ X_5 \end{pmatrix} - E\left[\begin{pmatrix} X_3 \\ X_5 \end{pmatrix} | X_1\right]\right)^T | X_1\right].$$