UTS

Stochastic Processes and Financial Mathematics

Lab/Tutorial 7

This lab is not assessed.

Question 1. MC path dependent option pricing

Consider the geometric Brownian Motion (GBM)

$$S_t = S_0 \exp\left((r - \sigma^2/2)t + \sigma B_t\right), \qquad 0 \leq t \leq T,$$

where $S_0 = 50$, $r = 3/100$, $\sigma = 4/10$ and $T = 5$ and B_t is a standard Brownian motion. Suppose that there are 250 trading days per year.

(a) Consider a European floating strike daily-monitored lookback call option with price at $t = 0$ given by

$$C = e^{-rT} E\left[\max\left(S_T - \min_{t \in \tau} S_t, 0\right)\right]$$

where $\tau = \left\{{}^{1}/_{250}, {}^{2}/_{250}, \dots, T\right\}$.

Taking $n = 10^5$, use Mathematica to calculate the crude Monte Carlo estimate

$$C_n = \frac{e^{-rT}}{n} \sum_{k=1}^{n} \max\left(S_T^{(k)} - \min_{t \in \tau} S_t^{(k)}, 0\right)$$

where $S_t^{(k)}$, $k \in \{1, \dots, n\}$, is a random sample of S_t. Also calculate the standard error of the estimated price.

Hint 1. Note that the GBM price equation at time $t + \Delta t$ is

$$\begin{aligned} S_{t+\Delta t} &= S_t \exp\left((r - \sigma^2/2)\Delta t + \sigma(B_{t+\Delta t} - B_t)\right) \\ &\overset{d}{=} S_t \exp\left((r - \sigma^2/2)\Delta t + \sigma B_{\Delta t}\right) \end{aligned}$$

where the last equality is in distribution (we are calculating an expectation so equality in distribution is sufficient). We see that the price $S_{t+\Delta t}$ can be calculated recursively from the previous day's price S_t.

Hint 2. Create a 10^5 by 1250 array of simulated Wiener increments $B_{\Delta t} \sim N(0, \Delta t)$.

Hint 3. Using a `For` loop or function `FoldList`, create an array of simulated stock prices for the initial price S_0 and each of the 1250 trading days using the recursive formula in Hint 1. Do this for $n = 10^5$ paths using the Wiener increments generated in Hint 2.

	$t = 0$	$t = 1/250$	$t = 2/250$	$\cdots$	$t = T = 5$
path $k = 1$	$S_0^{(1)}$	$S_{1/250}^{(1)}$	$S_{2/250}^{(1)}$	$\cdots$	$S_5^{(1)}$
path $k = 2$	$S_0^{(2)}$	$S_{1/250}^{(2)}$	$S_{2/250}^{(2)}$	$\cdots$	$S_5^{(2)}$
path $k = 3$	$S_0^{(3)}$	$S_{1/250}^{(3)}$	$S_{2/250}^{(3)}$	$\cdots$	$S_5^{(3)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
path $k = 10^5$	$S_0^{(10^5)}$	$S_{1/250}^{(10^5)}$	$S_{2/250}^{(10^5)}$	$\cdots$	$S_5^{(10^5)}$

Hint 4. For each path, use the Mathematica function `Min` to find the minimum stock price on each path and thereby the payoff on each path and proceed in the manner of Lab 6.

(b) Repeat the exercise in part (a) using R.

Question 2. Stationary Markov process

Let $B(t)$, $t \geq 0$, be a standard Brownian motion and

$$X_t = e^{-t} B(e^{2t}).$$

(a) Using the definition of covariance (not the specific form that covariance takes for a standard BM), find the mean function and show that covariance function

$$Q(t,s) = \operatorname{cov}(X_t, X_s) = e^{-|t-s|}.$$

Check your answer using definition of autocovariance for Brownian motion.

Hint 1. Begin with the case $s < t$.

Hint 2. Repeat for case $t < s$ and combine cases to show required result.

(b) Prove that X_t is strictly stationary.

Hint 3. See Chapter 4 Notes.

(c) Prove that X_t is Markov.

Hint 4. See Chapter 5 Notes.

(d) Find the distribution of the increment $\Delta X_t = X_{t+\Delta t} - X_t$.

(e) Determine if non-overlapping increments of X_t are independent.

Hint 5. Consider times $s < t < u$.